Chaos in Superstring Cosmology

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We show that the general solution of the Einstein-dilaton-antisymmetric-tensors field equations of all superstring theories exhibits a chaotic oscillatory behaviour of the Belinskii-Khalatnikov-Lifshitz type near a cosmological singularity. This result indicates that superstring cosmology is much more complex than is assumed in the scenarios presently discussed in the literature.

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Chaos plays a ubiquitous role in physics, at least in systems with sufficient complexity. Long ago, Belinskii, Khalatnikov and Lifshitz (BKL) discovered that the generic solution of the four-dimensional Einstein's vacuum equations near a cosmological singularity exhibits a never ending oscillatory behaviour [1]. This oscillatory behaviour has the character of a random process, whose chaotic nature has been intensively studied [2]. However, two results cast a doubt on the physical applicability, to our universe, of this chaotic picture. First, it was surprisingly found that the chaotic BKL oscillatory behaviour disappears from the generic solution of the vacuum Einstein equations in spacetime dimension $D \geq 11$, to be replaced by a monotonic Kasner-like power-law behaviour [3]. Second, it was proved that the general solution of the four-dimensional Einstein-scalar equations also exhibits a non-oscillatory, power-law behaviour [4], [5].

Recent developments in particle physics suggest that the long-range fields that can exist near a spacelike singularity (i.e., for energies above some symmetry-breaking threshold) are more numerous than the ones considered in the Standard Model, namely, the metric $g_{\mu\nu}$, some Yang-Mills fields and some Higgs fields. The most ambitious unified theory (and our best present candidate for a quantum theory incorporating gravity) is the theory of superstrings [6]. Superstring theory predicts that the massless degrees of freedom which can be generically excited near a cosmological singularity correspond to a high-dimension ($D = 10$ or 11) Kaluza-Klein-type model containing, in addition to Einstein's D-dimensional gravity, several other fields, which are scalars, vectors and/or forms (i.e., antisymmetric tensors). In view of the results quoted above, it is a priori unclear whether the full field content of superstring theory will admit, as generic cosmological solution, a chaotic BKL-like behaviour, or a monotonic Kasner-like one. Most of the string cosmology literature (notably the work on the pre-big-bang scenario [7]) has implicitly assumed a monotonic Kasner-like behaviour. Here, we report the result that the massless bosonic content of all superstring models ($D = 10$ IIA, IIB, I, het_E, het_SO), as well as of M-theory ($D = 11$ supergravity), generically implies a *chaotic BKL-like oscillatory behaviour* near[1] a cosmological singularity. It is the presence of various form fields that provides the crucial source of this generic oscillatory behaviour.

We consider a model of the general form

$$S = \int d^D x \sqrt{g} \left[R(g) - \partial_\mu \varphi \, \partial^\mu \varphi - \sum_p e^{\lambda_p \varphi} (d A_p)^2 \right] . \quad (1)$$

Here, the spacetime dimension D is left unspecified. We work (as a convenient common formulation) in the Einstein conformal frame. The integer $p \geq 0$ labels the various p-forms $A_p \equiv A_{\mu_1 \ldots \mu_p}$ present in the theory, with field strengths $F_{p+1} \equiv d A_p$, i.e. $F_{\mu_0 \mu_1 \ldots \mu_p} = \partial_{\mu_0} A_{\mu_1 \ldots \mu_p} \pm p$ permutations. The real parameter λ_p plays the crucial role of measuring the strength of the coupling of the dilaton to the p-form A_p. The model (1) is, as it reads, not quite general enough to represent in detail all the superstring actions. Indeed, it lacks additional terms involving possible couplings between the form fields. However, we have verified in all relevant cases that these additional terms do not qualitatively modify the BKL behaviour to be discussed below [8]. On the other hand, in the case of M-theory, the dilaton φ is absent, and one

[1]Our analysis applies at scales large enough to excite all Kaluza-Klein-type modes, but small enough to be able to neglect the stringy and non-perturbative massive states.

must cancel its contributions to the dynamics.

The leading Kasner-like approximation to the solution of the field equations derived from (1) is, as usual [1]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \simeq -dt^2 + \sum_{i=1}^{d} t^{2p_i(x)}\, (\omega^i)^2\,, \quad \varphi \simeq p_{\varphi}\,(x)\,\ln t + \psi\,(x)\,, \tag{2}$$

where $d \equiv D - 1$ denotes the spatial dimension, x stands for the spatial coordinates and $\omega^i\,(x) = e^i_j\,(x)\,dx^j$ is a time-independent d-bein. The spatially dependent Kasner exponents $p_i\,(x)$, $p_{\varphi}\,(x)$ must satisfy the famous Kasner constraints (modified by the presence of the dilaton):

$$p_{\varphi}^2 + \sum_{i=1}^{d} p_i^2 = 1\,, \quad \sum_{i=1}^{d} p_i = 1\,. \tag{3}$$

The set of parameters satisfying Eqs. (3) is (topologically) a $(d-1)$-dimensional sphere: the "Kasner sphere". When the dilaton is absent, one must set p_{φ} to zero in Eq.(3). In that case the dimension of the Kasner sphere is $d - 2 = D - 3$.

The approximate solution (2) is obtained by neglecting in the field equations for $g_{\mu\nu}$ and φ: (i) the effect of the spatial derivatives of $g_{\mu\nu}$ and φ, and (ii) the contributions of the various p-form fields A_p. The condition for the "stability" of the solution (2), i.e. for the absence of BKL oscillations at $t \to 0$, is that the inclusion in the field equations of the discarded contributions (i) and (ii) (computed within the assumption (2)) be fractionally negligible as $t \to 0$. As usual, the fractional effect of the spatial derivatives of φ is found to be negligible, while the fractional effect of the spatial derivatives of the metric contains, as only "dangerous terms" when $t \to 0$, a sum of terms $\propto t^{2g_{ijk}}$, where the *gravitational exponents* g_{ijk} $(i \neq j,\, i \neq k,\, j \neq k)$

are the following combinations of the Kasner exponents [3]

$$g_{ijk}(p) = 2\,p_i + \sum_{\ell \neq i,j,k} p_\ell = 1 + p_i - p_j - p_k\,. \tag{4}$$

The "gravitational" stability condition is that all the exponents $g_{ijk}(p)$ be positive.

It was shown in [3] that if the only stability conditions were the gravitational ones, they could be satisfied by the pure vacuum Einstein equations in $D \geq 11$, or, equivalently, by the dimensional reduction of these equations (without freezing any degree of freedom) in any lower dimension. We found, however, that this fact is crucially changed by the presence of form fields A_p. These fields give additional source terms on the RHS of the Einstein-dilaton field equations, thereby yielding further stability conditions. These stability conditions can be derived by solving, à la BKL, the p-form field equations in the background (2) and then estimating the corresponding "dangerous" terms in the $g_{\mu\nu}$- and φ-field equations. When performing this detailed analysis, one gets, as additional dangerous terms for $t \to 0$, a sum of "electric" contributions $\propto t^{2\,e^{(p)}_{i_1\cdots i_p}}$ and of "magnetic" ones $\propto t^{2\,b^{(p)}_{j_1\cdots j_{d-p-1}}}$. Here, the *electric exponents* $e^{(p)}_{i_1\ldots i_p}$ (where all the indices i_n are different) are defined as

$$e^{(p)}_{i_1\ldots i_p}(p) = p_{i_1} + p_{i_2} + \cdots + p_{i_p} - \frac{1}{2}\,\lambda_p\,p_\varphi\,, \tag{5}$$

while the *magnetic exponents* $b^{(p)}_{j_1\cdots j_{d-p-1}}$ (where all the indices j_n are different) are

$$b^{(p)}_{j_1\cdots j_{d-p-1}}(p) = p_{j_1} + p_{j_2} + \cdots + p_{j_{d-p-1}} + \frac{1}{2}\,\lambda_p\,p_\varphi\,. \tag{6}$$

To each p-form is thus associated a double family of "stability" exponents $e^{(p)}$, $b^{(p)}$. This generalizes the discussion of [9] on the effect of vector fields in $D = 4$. The condition for the stability of the Kasner-like solution (2), i.e. the condition for the absence of BKL oscillations, is that *all* the exponents $g_{ijk}(p)$, $e^{(p)}_{i_1 \ldots i_p}(p)$, $b^{(p)}_{j_1 \ldots j_{d-p-1}}(p)$ (considered for all possible indices i, j, k, i_n, j_n, and all possible forms) be strictly positive for the (spatially varying) values of the Kasner parameter $p_\alpha(x)$ involved in Eqs. (2), since this would imply that all dangerous terms are fractionally negligeable as $t \to 0$.

The main result reported here is that, for all superstring models, there exists no open region of the Kasner sphere where all the stability exponents $g(p)$, $e(p)$, $b(p)$ are strictly positive. Accordingly, the generic solution of the low-energy string models can never reach a monotonic Kasner-like behaviour. This result has been obtained by (i) a direct algebraic analysis of the stability conditions for M-theory, and for the heterotic model, and (ii) a crucial use of the various string dualities to transfer the applicability of the analysis of (i) to the other superstring models. [E.g., we use the T-duality between IIA and IIB superstring theories to define a map of the corresponding Kasner parameters, $p^{IIA} = \pi(p^{IIB})$, which exhibits the equivalence of the Kasner-stability conditions of the two models.]

Following the BKL approach [1], we have then gone further and studied the evolution of the fields near a cosmological singularity as a sequence of Kasner-like "free flights" interrupted by "collisions" against the "potential walls" corresponding to the various stability-violating exponents g, e or b. We found a universal "collision law" giving the Kasner exponents of the Kasner epoch following a collision in terms of the old ones [8]. It generalizes the collision law obtained in four dimensions, which is known to define a chaotic discrete dynamics [2].

Consequently, in all string models, the general solution near a cosmological singularity for the massless bosonic degrees of freedom exhibits BKL-type oscillations, i.e. a (formally infinite) alternation of Kasner-epochs. This fact might have a significant impact on the pre-big-bang scenario [7] which strongly relies on the existence, near a (future) cosmological singularity, of relatively large, quasi-uniform patches of space following a monotonic, dilaton-driven Kasner behaviour. By contrast our findings suggest that the *spatial inhomogeneity* continuously *increases* toward a singularity, as all quasi-uniform patches of space get broken up into smaller and smaller ones by the chaotic oscillatory evolution. In other words, the spacetime structure tends to develop a kind of "turbulence" [10]. This result indicates that superstring cosmology is much more complex than is assumed in the simplified models currently discussed in the literature.

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