Planetary Distances According to General Relativity

by

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Abstract. The corrections to the radius vector of a planet, calculated according to the conventional Newtonian theory, that are required by general relativity are discussed. For Mercury the principal effect is one of anomalistic period and coefficient 2 km. If the effects smaller than this were detectable it would constitute a fourth observational test of general relativity.

Now that interplanetary distances are being inferred with very high nominal precision from the to-and-fro time of travel of radar signals, it becomes worthwhile to enquire what corrections to the conventional values calculated according to Newtonian theory may be required by general relativity, in addition to the well-known effects of the secular advance of the perihelion. W. de Sitter (1916) has developed the astronomical consequences of general relativity with all the rigor that is desirable for the present application. But in treating the motion of a planet he uses the method of variation of arbitrary constants, giving explicit expressions for the major axis, mean longitude at epoch, eccentricity, and longitude of perihelion. The method, while it is superior to any other for exhibiting the well-known secular advance of the perihelion in a simple and direct way, is not so ready for my purpose here, which is to obtain explicit expressions for the orbital longitude and the radius vector. It is possible, of course, to obtain the two coordinates mentioned by transformation of de Sitter's expressions for the four elements. But here I desire to fix the constants of integration so that the calculated orbital longitude will agree as nearly as may be with the value obtained by direct observation. The attempts I have made to impose the necessary conditions on de Sitter's developments have led to prolix expressions that are difficult to interpret metrically. It is easier to proceed directly to the calculation of the coordinates by Hansen's method, and that is what I have done here.

In order to save space I omit most of Hansen's formulas, which can be found in many places, for example in Brouwer and Clemence (1961). I also omit demonstrations of those propositions of relativity that can be found either in de Sitter (1916) or in the standard works, especially Pauli (1958).

In the theory of relativity itself, four conditions are left unspecified. Only six of Einstein's ten equations of gravitation are independent; four relations of identity exist among them.

If we neglect the mass of the planets, and assume that the gravitational field of the sun is static and spherically symmetric, then the three components g_{14}, g_{24}, g_{34} of the fundamental tensor all vanish, which fixes three of the four conditions at our disposal. The assumption that the field is static (not varying with the time) is equivalent to specifying how the time is to be measured, and in fact is equivalent to saying that the solar system is neither expanding nor contracting (neglecting of course any possible effects from the planets and from loss of mass by solar radiation). Whether either (or both) of our practical measures of time (ephemeris time or atomic time) accords with the assumption of a static field is a matter for experimental verification; no discrepancy has yet appeared, nor is likely to until distance measurements in the solar system have been continued for some time.

It follows from the assumption of spherical symmetry that the motion of a planet is in a plane passing through the center of the sun.

Under the two assumptions the square of the four-dimensional line element may be written $ds^2 = -(1+\alpha)dr^2 - (1+\beta)r^2 d\theta^2 + (1+\gamma)c^2 dt$ where r is the radius vector, θ the true orbital longitude, c the velocity of light regarded as a constant of the theory, t the time, and α, β, γ three small parameters which may be developed as power series in λ/r, λ being the gravitational radius of the sun, 1.48 km. (My λ is called m by Pauli and λ_0^2 by de Sitter.) It suffices to take α and β to the first order and γ to the second, γ being needed to one higher order because it is multiplied by c^2.

Now since the manner of measuring the time has already been specified, and since θ is a conventional coordinate, it is evident that the fourth condition, which remains at our disposal, will fix the measure of r. It also follows that once this condition is fixed, the values of α, β, γ must become determinate, and indeed it turns out that the equations of motion just suffice for their determination.

Of the numerous conditions that may be imagined, only two have gained much popularity in the study of planetary motion, probably because, with the exception of the one giving the line element of special relativity, they are the only possible ones in a static homogeneous universe; and while the universe may not be static, there is no objection to supposing the gravitational field of the sun to be so, which is sufficient for my purpose.

The two conditions are:

A. $\beta = 0$

B. $\alpha = \gamma_1 = 0$

where γ_1 is the first-order part of γ . They lead to the values

A. $\beta = 0, \; -\alpha = \gamma = -2\lambda/r$

B. $-\alpha = -\beta = \gamma_1 - 2\lambda/r, \quad \gamma = -2\lambda/r + 2\lambda^2/r^2$

In system A the second-order portion of is identically zero. System A corresponds (to the stated accuracy) to the line element of K. Schwarzschild (also used by Einstein), in which rigorously

$$1 + \alpha = 1/(1 - 2\lambda/r)$$
$$\beta = 0$$
$$\gamma = -2\lambda/r$$

In system A the measured velocity of light is

A. $v = c\left[1 + \frac{1}{2}\gamma\left(1 + \cos^2 V\right)\right]$

where V is the angle between a light-ray and the radius vector; since γ is negative, v is a maximum in directions perpendicular to the radius vector.

In system B the line element was first given by de Sitter. It corresponds (to the stated accuracy) to the one afterwards given by H. Weyl, in which rigorously

$$1 + \alpha = 1 + \beta = (1 + \lambda/2r)^4$$
$$1 + \gamma = (1 - \lambda/2r)^2/(1 + \lambda/2r)^2$$

In system B the coordinates are isotropic: that is, at any point the velocity of light is the same in all directions. Its measured value is

$$\text{B.}\quad v = c(1+\gamma)$$

There is no inconsistency in supposing the measured velocity of light to be less than c, or to depend on direction. In system B we must suppose that distances are measured with rigid rods, which are contracted by the gravitational acceleration when they are pointed toward the sun; the propagation of light being correspondingly slower in that direction (as viewed from outside the system), its measured value remains the same in all directions. In system A we must suppose that distances are inferred from the travel-time of the light itself, and no such compensating effect takes place. System A is then the one appropriate to the present purpose, but I make the calculations for both, in order to show the differences between them. The gravitational red shift (to the first order), the deflection of light-rays passing near the sun, and the advance of the perihelia are all the same in both systems.

In system A Kepler's third law is exact for a circular orbit, whereas in system B the corresponding law is

$$\text{B.}\quad n^2a^3 = k^2m\,(1-3\lambda/a)$$

In system A the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 + \mu/r^2 = \lambda(-2\dot{\theta}^2 + 3\dot{r}^2/r^2 + 2\mu/r^3),$$

$$\frac{d}{dt}(r^2\dot{\theta}) = \lambda(2\dot{r}\dot{\theta}),$$

the dot standing for differentiation with respect to the time. In system B they are

$$\ddot{r} - r\dot{\theta}^2 + \mu/r^2 = \lambda(-\dot{\theta}^2 + 3\dot{r}^2/r^2 + 4\mu/r^3),$$

$$\frac{d}{dt}(r^2\dot{\theta}) = \lambda(4\dot{r}\dot{\theta}),$$

where $\mu = n^2a^3 = c^2\lambda$, n is the mean motion, and a is half the major axis. If the right-hand sides of the equations are put equal to zero we have the ordinary Newtonian equations of elliptic motion. Hence, since λ is a very small parameter, the complete equations may be solved by any of the conventional methods of the planetary theory, treating the right-hand sides as if they were the partial derivatives of the ordinary disturbing function with respect to r and θ .

In order that the results may be applicable to any planet, I make the developments in powers of e, the eccentricity, neglecting powers above the second, and I take ℓ , the mean anomaly, as the independent variable. Then we have for the four components of the right-hand sides of the equations.

1. $r\dot{\theta}^2/n^2 a = 1 + \frac{1}{2} e^2 + 3e \cos \ell + \frac{9}{2} e^2 \cos 2\ell$

2.

3.

4.

Hansen's multipliers A and B (not to be confused with the letters denoting two different measures of the radius vector) are

A =

B =

where denotes an which is held constant in integration or differentiation.

Hansen's T is given by

which is equivalent to

Working with the last formula, and being careful to adjust the constants of integration so that , the perturbation of the mean longitude, contains no constant term, no term proportional to the time, and no term strictly proportional to sin ,
I find for the four perturbations corresponding to the four numbered expressions above,

1.
2.
3.
4.
1.
2.
3.
4.

It is noteworthy that only three constants of integration appear in the developments as a consequence of the radial "forces" containing only cosines and the tangential "force" only sines; the corresponding physical condition is that the results are valid no matter where the perihelion may happen to be, as must necessarily be the case in a spherically symmetric field.

Multiplying the four portions by , and by -2, +3, +2, +2 for case A, and by -1, +3, +4, +4 for case B we obtain for the final perturbations

A.
B.
A.

B.

All of the developments have the property that the lowest power of e in a coefficient is the same as the multiple of in the argument, unless it vanishes as it does in the constant part of for system A, and that only odd powers of e are associated with odd multiples of , and only even powers with even multiples. Thus in every case the lowest neglected power of e is two higher than the highest one shown.

It is easily shown (see Clemence 1946) that the portions of the perturbations factored by nt are precisely equivalent to a secular advance of the perihelion amounting to for both systems A and B, a well-known result. For the other terms, neglecting the cube and higher powers of e, we may put , the perturbation of the true orbital longitude, for , and , the perturbation of the radius vector, for . If we then express the coefficients in seconds of arc for and in kilometers for we obtain

A.

B.

where a is measured in astronomical units, and

A.

B.

for the corrections to the Newtonian values of the coordinates.

The corrections to the longitude are too small to be detected in the present state of astronomy. As to the radius vector, if it were possible to measure the distance to Mercury with a precision of a kilometer for a single measurement, and to continue the measurements over several revolutions of the planet, it should be possible to detect the principal periodic term. Such detection could not, however, be regarded as a test of general relativity, as Dirk Brouwer has pointed out to me, because a small correction to the eccentricity of the orbit would have precisely the same effect on the radius vector, and I have ascertained that the amount is well within the uncertainty in the value of the eccentricity determined by optical observations. It follows that radar observations of the precision mentioned would determine the eccentricity (and the longitude of the perihelion) with considerably greater accuracy than is attainable by any other method, not only for Mercury, but for any planet to which the method can be applied. The relativity-effects then would have to be sought in the terms factored by the square of the eccentricity, which amount at most to half a kilometer.

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