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How the Jones Polynomial gives rise to physical states of quantum General Relativity

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Solutions to both the diffeomorphism and the hamiltonian constraint of quantum gravity have been found in the loop representation, which is based on Ashtekar's new variables. While the diffeomorphism constraint is easily solved by considering loop functionals which are knot invariants, there remains the puzzle why several of the known knot invariants are also solutions to the hamiltonian constraint. We show how the Jones polynomial gives rise to an infinite set of solutions to all the constraints of quantum gravity thereby illuminating the structure of the space of solutions and suggesting the existance of a deep connection between quantum gravity and knot theory at a dynamical level.

An important step in the canonical quantization of 4-dimensional general relativity is the construction of the space of physical states, that is the space of wavefunctions which are annihilated by the diffeomorphism and hamiltonian constraint operators. The discovery of the Ashtekar variables for canonical 3+1 dimensional general relativity [1] has led to the construction of a loop representation for quantum gravity, in which wavefunctions are functionals of loops [2]. In this context the diffeomorphism invariance is quite elegantly described by knot theory. Since knot invariants are diffeomorphism invariant loop functionals, one is naturally led to consider knot invariants as candidates for the states of quantum gravity.

This approach is reminiscent of the canonical quantization in the geometrodynamical variables, where the diffeomorphism constraint is formally solved by choosing the states to be functionals of three-geometries. The hard part is to solve the hamiltonian constraint (the Wheeler-Dewitt equation). Since this constraint encodes the dynamical evolution of the Einstein equations one does not expect that notions from knot theory are going to be helpful for finding solutions. The main intention of this essay is to show that, quite surprisingly, there exists a connection between knot theory and quantum gravity *also at the dynamical level.* To be more specific, we will exhibit a solution to all the constraints of quantum gravity with a cosmological constant in the loop representation related to the Jones polynomial, and we will indicate how this one solution gives rise to an infinite new set of vacuum solutions.

It was only quite recently that techniques for concrete calculation of the action of the constraints in the loop representation were introduced. Firstly, through the introduction of a set of "coordinates" on loop space [3] and the advances in the understanding of Chern-Simons theories [4, 5], it was possible to give an explicit, analytic form for several knot invariants. Secondly, the understanding of the action of the constraints of quantum gravity in the loop representation has been greatly enhanced when they were expressed in terms of differential operators in loop space [6, 7]. It was through these developments that we were in a position to explicitly apply the diffeomorphism and hamiltonian constraints in the loop representation to some of the known analytic knot invariants in search for solutions with triple self-intersections. (A general argument shows that only in this case one can obtain a nondegenerate metric [8].)

A positive result was achieved some time ago, when we showed that the second coefficient of the Alexander-Conway knot polynomial (a knot invariant associated with the classic Arf and Casson invariants) was annihilated by all the constraints of quantum gravity even when considered on knots with triple self-intersections [9]. In spite of the appeal of this result, which pointed out a remarkable connection between gravity and knot theory at a dynamical level and for the first time exhibited a concrete physical state of the quantum gravitational field, it was also true that this one example did not shed much light on the structure of the space of solutions. The rest of this essay is devoted to show that this puzzling result emerges quite naturally from simple notions of knot theory.

The new canonical variables for general relativity introduced by Ashtekar are a Lie-algebra valued connection $A_a(x)$ and a triad $E^b(y)$ on a three-manifold Σ, while the three-metric on Σ becomes a derived quantity, $q^{ab} = \mathrm{Tr} E^a E^b$. There are two representations of quantum gravity based on the new variables: the connection representation in which wavefunctions are functionals of the Ashtekar connection [10], and the loop representation in which wavefunctions are functionals of loops [2]. At a heuristic level, these two representations can be related by a "loop transform" (see below) which maps states and operators in the connection representation to the loop representation, in analogy to the Fourier transform which in quantum mechanics relates the position representation to the momentum representation. Loop representations have been used for several theories, including Maxwell electrodynamics [11, 12], 2+1 gravity [13], and even for Yang Mills calculations on the continuum [14] and the lattice [15, 16]. Far from being a mathematical nicety, they are a quite powerful and concrete way of analyzing the quantum dynamics of theories based on a connection.

Let us recall that in the connection representation of quantum gravity based on Ashtekar variables there exists a state that is a solution to all the constraints given by [17, 18],

$$\Psi_\Lambda[A] = \exp(-\frac{6}{\Lambda}\int_\Sigma \tilde{\eta}^{abc}\mathrm{Tr}[A_a\partial_b A_c + \frac{2}{3}A_a A_b A_c]). \tag{1}$$

That is, the exponential of the Chern-Simons form constructed from the Ashtekar connection is a solution to all the constraints of quantum gravity with a cosmological constant $\Lambda \neq 0$ in terms of the Ashtekar new variables. Moreover, the determinant of the three metric has a definite nonzero value on this state.

Having such a state in the connection representation it is natural to ask the question if it has a counterpart in the loop representation. Since the loop transform

$$\Psi[\gamma] = \int \mathcal{D}A \ \ \mathrm{Tr}(\mathrm{P}\exp\oint \dot{\gamma}^a A_a)\Psi[A] \tag{2}$$

is just a formal entity at present (we do not know how to perform the integral on the right) one simply does not know in general how to find the counterpart in the loop representation for a given state. However for the particular state $\Psi_\Lambda[A]$, the loop transform turns out to be identical to the expression for the expectation value of the holonomy in a Chern-Simons theory. This can readily be seen by replacing the expression for $\Psi[A]$ in (2) with $\Psi_\Lambda[A]$. It turns out that this expression has been evaluated by various techniques in the context of Chern-Simons theories [4, 5]! The transform $\Psi_\Lambda[\gamma]$ is a knot polynomial which is closely related to the Jones polynomial. For the particular case considered $\Psi_\Lambda[\gamma]$ is a polynomial in Λ where each coefficient is a knot invariant depending on γ. This result can be generalized to the case of intersecting loops [18].

One can therefore conclude that — formally by construction — $\Psi_\Lambda[\gamma]$ is a solution to all the constraints of quantum gravity in the loop representation with a cosmological constant, and it is nondegenerate if one considers intersecting loops. A relevant question is therefore: since we have the appropriate technology to apply the constraints of quantum gravity in the loop representation to concrete knot invariants, can we actually show that this polynomial is a solution to the constraints? The answer is yes and the result is surprising.

Since we are dealing with knot invariants, we will not be concerned with the diffeomorphism constraint [19]. The non-trivial constraint to satisfy is the hamiltonian. With a cosmological constant it can be written as

$$\mathcal{H}_\Lambda = \mathcal{H}_0 + \Lambda \mathrm{det} q \tag{3}$$

where $\mathcal{H}_0$ is the vacuum hamiltonian constraint and $\mathrm{det}q$ is the determinant of the three metric. We now write $\Psi_\Lambda[\gamma]$ explicitly as a polynomial in Λ:

$$\Psi_\Lambda[\gamma] = c_0[\gamma] + c_1[\gamma]\Lambda + c_2[\gamma]\Lambda^2 + \ldots \tag{4}$$

The coefficients $c_i[\gamma]$ correspond to concrete analytical expressions. For instance, $c_1[\gamma]$ is given by the celebrated expression of Gauss for the self-linking number of γ,

$$c_1[\gamma] = \oint \oint ds dt \; \dot\gamma^a(s)\dot\gamma^b(t)\epsilon_{abc}\frac{\gamma^c(s)-\gamma^c(t)}{|\gamma(s)-\gamma(t)|}. \tag{5}$$

This expression is finite despite appearance [20]. $c_0[\gamma]$ is 1 when the loop has one connected component and 0 else. $c_2[\gamma]$ can be decomposed as $c_2[\gamma] = c_1[\gamma]^2 + \rho[\gamma]$, where $\rho[\gamma]$ is a well known knot invariant, the second coefficient of the Conway polynomial, also related to the Arf and Casson invariants [5].

Applying the hamiltonian constraint operator based on [6, 7] to the polynomial we obtain again a polynomial in Λ each of whose coefficients should vanish independently. The fact that the hamiltonian constraint has a homogeneous term and a term linear in Λ means that different orders in the coefficients of $\Psi_\Lambda[\gamma]$ will get mixed. As a final result of the calculation it turns out that several terms combine to cancel provided some conditions are fulfilled. These conditions turn out precisely to require that some portions of the coefficients be annihilated by the vacuum constraint. The first one is

$$\hat{\mathcal{H}}\, c_0[\gamma] = 0, \tag{6}$$

which is immediate to prove [21]. More important is that one finds

$$\hat{\mathcal{H}}\, \rho[\gamma] = 0. \tag{7}$$

That is, $\rho[\gamma]$, the second coefficient of the Conway polynomial has to be annihilated by the vacuum hamiltonian constraint! Moreover, it is not annihilated by the determinant of the three metric, and actually this was crucial for the cancellation of other terms in the computation. Hence we have arrived in a simple and conceptual way at the result first introduced in [9], where it was obtained through a laborious computation, that the second coefficient of the Conway polynomial, a knot invariant, is a nondegenerate solution to the hamiltonian constraint of quantum gravity.

In fact one can consider the above calculation for higher order coefficients thereby systematically exploring the structure of these states, and similar results can be obtained: for each order in the polynomial there appear portions of the coefficients that are annihilated by the vacuum hamiltonian constraint and not by the determinant of the metric. The Jones polynomial therefore emerges as an infinite tower of physical states of vacuum quantum gravity. This suggests that a deep and beautiful connection exists between quantum gravity and knot theory at a level that transcends pure diffeomorphism invariance and takes into account the full dynamics of general relativity.

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there are delicate regularization issues that have to be analyzed in more detail.

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