ANALYSIS OF THE ROTATING FLAT PLATE NEWTONIAN GRAVITATIONAL CONSTANT EXPERIMENT*

by

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ABSTRACT

This essay covers one portion of our program to find a new experimental method of improving our knowledge of the Newtonian gravitational constant (G). According to the NBS Technical News Bulletin (October 1963) the presently accepted value is 6.670 ± 0.015 x 10^{-11} $m^3kg^{-1}sec^{-2}$ (three standard deviations). The rotating flat plate experiment utilizes the gravitational interaction between two optically flat and parallel rectangular solids, one of which is rotating at constant speed and the other in a resonant mount. The analysis establishes that the gravitational interaction between the two plates is a second order gravitational gradient and that the dynamic interaction will be at twice the rotation frequency of the rotating plate. The magnitude of this gravitational gradient is of the order of 10^{-8} sec^{-2} and depends only on the density of the plate for fixed dimension ratios. Similar experiments have already been carried out at the Hughes Research Laboratories, and it has been found possible to eliminate all external sources of dynamic noise from the detecting system except for the internal thermal noise. An error analysis has been carried out on all the primary system parameters to determine their required precision. Most of the technology required in this experiment has been developed in previous gravitational experiments. We conclude that if the interferometer techniques are successful and the noise isolation techniques can be extended so that the instrument noise is predominantly thermal noise, accuracies approaching one part in 10^6 should be obtained in the measurement of the gravitational constant. To reach this level of accuracy requires an experiment time of half a day. The primary limitations of this experiment will be density inhomogeneities in the plates, the stability of the mechanical damping constant, and the nonlinearities and drift in the suspension system.

A. INTRODUCTION

We propose a dynamic Cavendish experiment designed to improve the accuracy of our knowledge of the Newtonian gravitational constant (G) to one part in 10^6. The gravitational constant currently is known to one part in 500 (three standard deviations). To date, the most accurate of the various experiments to determine G is the "time of swing method" of Heyl.[1] This experiment consists of two concentric torsion balances and is similar to the Cavendish apparatus (see Fig. 1). One balance is held stationary while the other is excited into a torsional oscillation. When the two balances are aligned in parallel, the period of swing is less than when they are aligned at right angles. In the former position, the gravitational attraction between the two balances adds to the torsional spring restoring force; in the latter position, it subtracts from it. The gravitational constant is obtained from measurement of the difference in periods between the near and far positions. The periods were on the order of a half hour, and could be measured to 0.1 sec.

A method of determining G to higher accuracy currently is being tested at the University of Virginia.[2] This experiment is designed to improve the knowledge of G to one part in 10^4; with future versions, accuracies greater than one part in 10^5, and possibly one part in 10^6, should be allowable. It also consists of two concentric torsional balances. One balance is free to rotate under the attraction of the second, while the second is motor-driven and servo-controlled to maintain constant angular position with respect to the first. Hence, both balances will rotate through 360°, while a constant torque is being maintained on the free balance. The angular displacement, after many hours, determines the gravitational constant.

In trying to push the determination of the gravitational constant to higher accuracy, all of the various possible experiments ultimately approach the limitation of precision in determining the relative position of the masses in the system and the homogeneity of density within the masses themselves. In addition, many experiments are affected by spurious nongravitational forces or stray gravitational effects of nearby masses which are difficult to separate from the desired gravitational interaction forces between the source masses and the detection masses.

In an attempt to overcome some of these problems, we propose a dynamic Cavendish experiment design which utilizes the gravitational interaction between two optically flat and parallel rectangular solids. The generator plate is rotated at a constant speed, and the detector plate is allowed to vibrate on a mount that is resonant at twice the rotation frequency of the generator. The principle of operation is basically that used by Forward and Miller[3] (see Appendix) The detecting structure proposed is a plate with a mass quadrupole moment on a resonant torsional suspension. This type of detector is similar to that used in Zahradnicek's resonance experiment.[4]

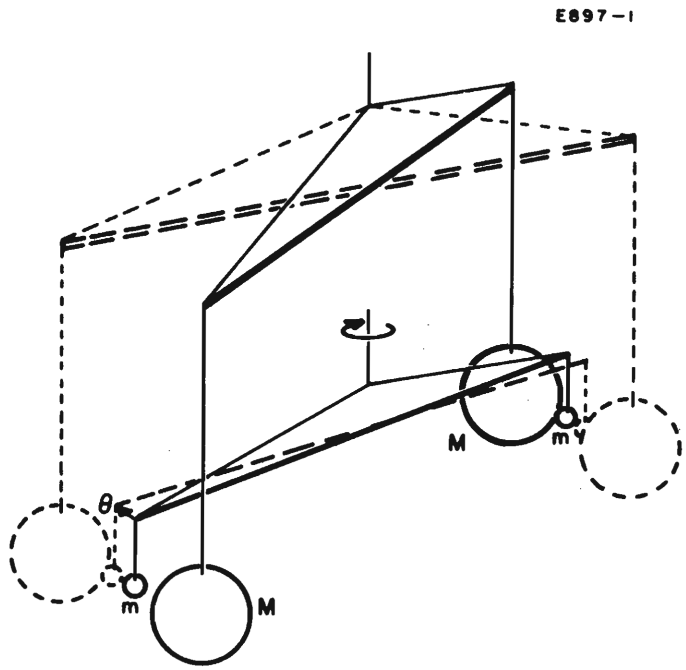

Fig. 1. Cavendish type coaxial torsion balance system.

Zahradnicek's apparatus consists of the usual two coaxial torsion balances of the Cavendish type. One balance is excited into simple harmonic motion at the same frequency as the resonant frequency of the second balance. In both the proposed experiment and Zahradnicek's experiment, the resonant response of the second balance builds up in amplitude many times that of an equivalent static Cavendish system, thus allowing more accurate measurements. Also, both methods are not susceptible to errors due to the presence of nearby stationary objects. Since the detector is a resonant system, the response depends only on the generator masses which are moving at the proper frequency.

The advantages of the proposed experiment over Zahradnicek's experiment are: (1) the relative position of all the interacting masses can be measured accurately with interferometer techniques (even during the experiment, if required); (2) flat optically clear plates can be machined to greater precision dimensionally, and can be controlled with respect to density homogeneity, and (3) the generating masses are moving at half the frequency of the detector resonant response and, therefore, predominantly all of the nongravitational noise produced by the generator is at a frequency which is outside the detector response frequency.

B. SYSTEM DESCRIPTION

The system design that we proposed to utilize is shown schematically in Fig. 2. The heart of the system is two precision machined plates which have optically flat and parallel sides. The upper plate is suspended by a torsional mount, whose stiffness is determined very precisely by prior independent measurement of the torsional natural frequency. Such techniques of stiffness determination are well established, since they are used in all of the important gravitational constant experiments described in Section A. The accuracy required is discussed in Section D. The torsional fiber is subsequently mounted to a platform which is vibrationally isolated from external disturbances. The lower plate is connected to the rotor of a magnetically suspended rotor, which is driven at constant angular speed by a servo-controlled motor.

Precision alignment of the two plates is obtained by interferometer techniques. Using these techniques, distances can be established to an accuracy of 5×10^{-7} cm (Ref. 5).

Measurement of angular displacement can be accomplished by recording the deflection of a light beam off a mirror mounted at the center of the resonant plate. These methods also have been used in the experiments cited above. In particular, the apparatus developed by Dicke and Roll[6] gives an accuracy of 10^{-9} rad. In that experiment, the source of light for the optical detector was a flashlight bulb. The light was focused through a narrow slit, reflected off the torsion balance, E897-2

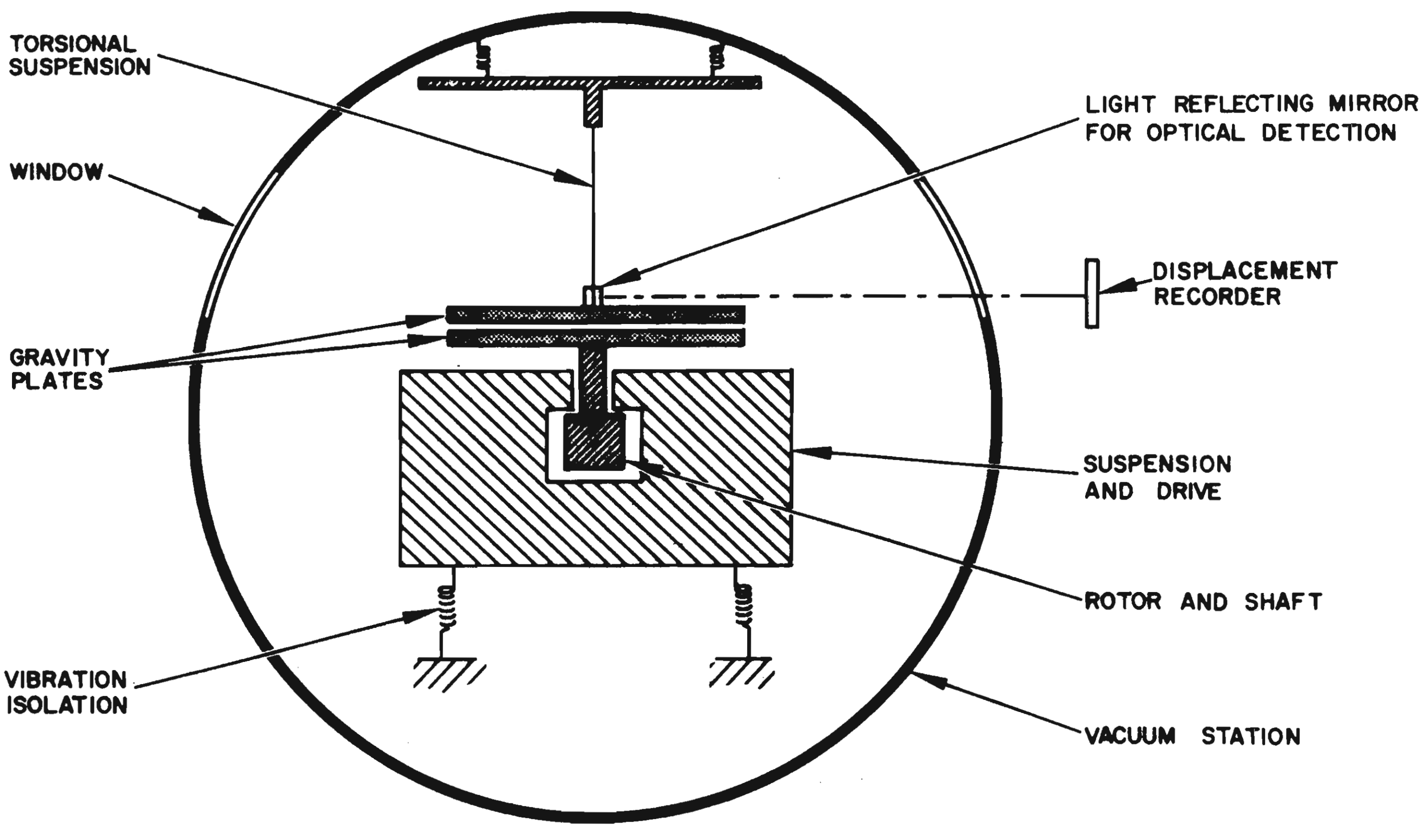

Fig. 2. System schematic.

and focused again on a very narrow wire. The wire was made to oscillate at its resonant frequency, and a photomultiplier was placed behind it. When the diffraction pattern centered exactly on the equilibrium position of the oscillating wire, the photomultiplier detected only the even harmonics of the wire fundamental frequency. When the torsion balance rotated slightly, shifting the diffraction pattern off center, the fundamental frequency became detectable in the photomultiplier output. The amplitude of the fundamental is proportional to the angular displacement for small angles.

The entire system, including motor, is enclosed in a vacuum system. This is necessary to avoid acoustic coupling from external sources, and also to avoid air currents.

It is obvious that in order to detect the very weak dynamic gravitational forces being generated by the rotating plate, the generator and detector must be well shielded to prevent acoustic and electromagnetic coupling. The detector is highly sensitive to acoustic noise with a frequency component at its resonance frequency, but experience has shown that the acoustic noise can be eliminated by placing the detector in a vacuum chamber at a few milliTorr.

Although an ideal detector is theoretically insensitive to vibrations of the mounting structure, in practice a small amount of the vibrations in the mount leaks into the sensing mode. Therefore, an effort must be made to keep the detector-mount vibrations at a low level. This is accomplished by suspending the detector in the chamber with springs. The generator is isolated from the workbench by compression springs.

Electromagnetic coupling can occur by direct interaction of the rotating magnetic fields of the motor with the arms of the detector. Direct coupling of the rotating magnetic fields is eliminated by using a phase-locked asynchronous drive. In this mode of operation of the generator, the generator motor is driven by currents at some higher frequency so that they do not excite the detector resonant mode. The amplitude of the drive voltage is controlled by a servo loop so that the rotor remains at a constant speed. The servo loop can be made so tight that both the frequency and the phase of the rotor can be held tightly to the phase of a reference signal from a precise oscillator.[3]

Dynamic gravity devices which are similar to this experiment in many ways have been operated successfully at Hughes Research Laboratories[3] (see Appendix). These devices have been perfected to the sensitivity level of thermal noise; all other sources of external disturbances have been reduced to less than this. Because of the similarities in design, we expect that with sufficient effort the noise disturbances of the flat-plate Cavendish experiments also will be determined predominantly by thermal noise.

C. GRAVITATIONAL INTERACTION

The determination of G requires a precise calculation of the gravitational interaction of the two plates. This calculation involves integrating the effect of each element of mass of the rotating plate on the elements of the resonant plate. The tangential component of the gravitational force gives the torsional coupling, from which the response (and therefore G) is determined.

The configuration for calculating the gravitational interaction is shown in Fig. 3. The gravitational potential $\mathcal{G}$ at r due to all mass in V' is

$$\mathcal{G} = \int_{V'} \frac{G\rho \, dV'}{|r - r'|} \tag{1}$$

where r' is a function of θ as the plate rotates.

The tangential component of the gravitational force is given by

$$dF = \left[\nabla \int \frac{G\rho \, dV'}{|r - r'|} \right] \cdot i\hat{\theta} \tag{2}$$

where $\hat{\theta}$ = unit vector in direction of rotation.

The integral over all r on the plate gives the total interaction:

$$F = G\rho \iint \nabla \left| \frac{1}{r - r'} \right| dV' \, dV \quad . \tag{3}$$

In this paper we do not propose to carry out the integration indicated in eqs. (1) through (3). We shall assume that the mathematics may be carried out with any desired degree of accuracy, and that the experiment will be limited only by laboratory measurement.

In order to estimate the gravitational interaction of the two plates, we shall consider only their gravitational quadrupole moments. Thus, we lump the mass of each as shown in the configuration of Fig. 4.

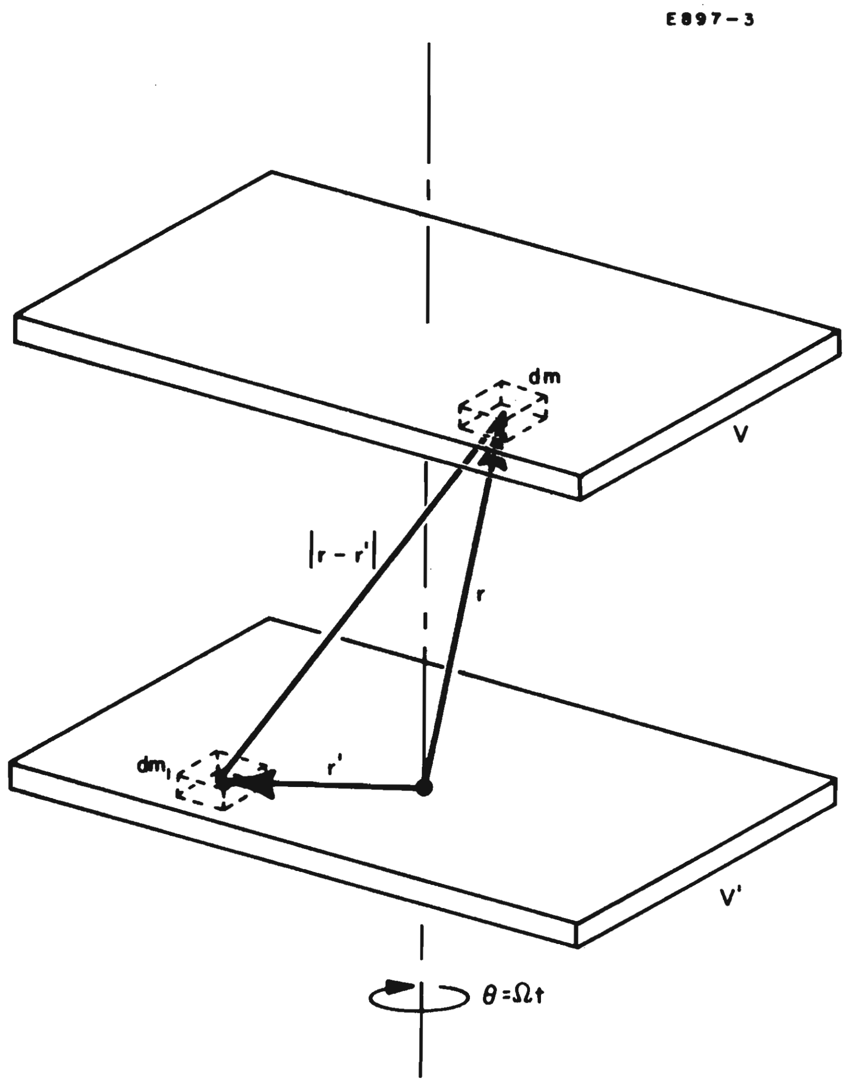

Fig. 3. Calculation of gravitational force by integration over all elements and taking tangential components.

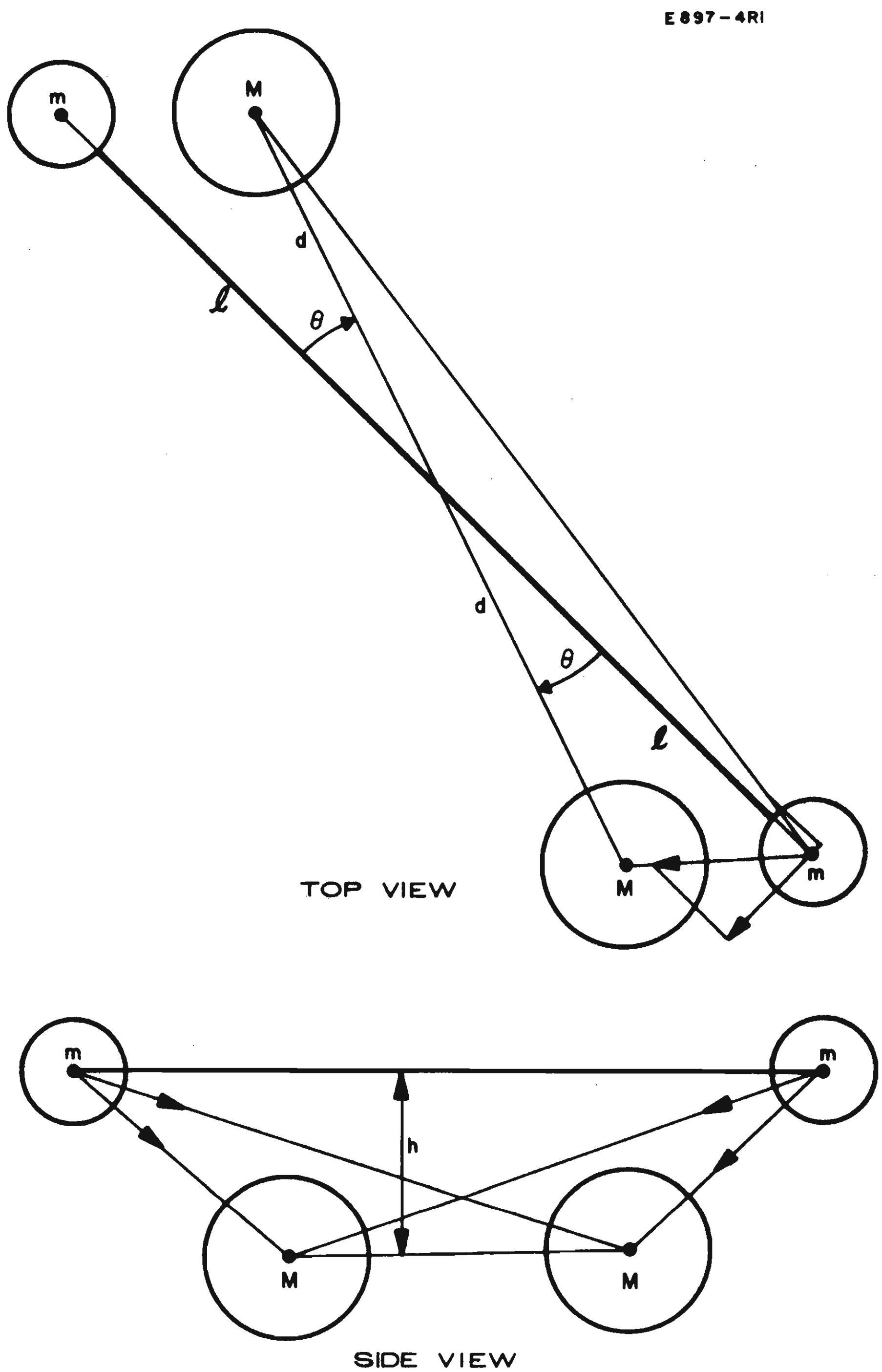

Fig. F-4. Model for gravitational-interaction calculation.

In the Appendix, it has been shown that a quadrupole gravitational detector is excited by a quadrupole generator. The excitation is given by

$$F = (GMm\,d/R^3)\left[6\Lambda\cos\theta + 35\Lambda^3\cos^3\theta + \frac{693}{4}\Lambda^5\cos^5\theta + \frac{6435}{8}\Lambda^7\cos^7\theta + \ldots\right]\sin\theta \qquad (4)$$

where $\Lambda = \ell d/R^2$; $R = \ell^2 + d^2 + h^2$; M = generator mass, m = detector mass. $\theta = \Omega t$ is the rotational angle, for constant angular frequency Ω. To simplify, we invoke the trigonometric identities:

$$8\cos^3\theta\sin\theta = \sin 4\theta + 2\sin 2\theta$$

$$32\cos^5\theta\sin\theta = \sin 6\theta + 4\sin 4\theta + 11\sin 2\theta$$

$$\vdots$$

$$\text{etc.} \qquad (5)$$

Substituting these into (4) yields

$$F = (GMm\,d/R^3)\,3\Lambda\left[1 + \frac{35}{12}\Lambda^2 + \frac{1155}{128}\Lambda^4 + \ldots\right]\sin 2\Omega t \quad . \qquad (6)$$

We have dropped the higher order terms in Ω, keeping only the lowest at 2Ω. Hence, the dominant driving force on the detector will be at a frequency which is twice the generator rotation frequency. For very close distances between generator and detector, h is small compared with d and ℓ, and we have as the limiting value of Λ (for ℓ = d):

$$\Lambda = \ell d/R^2 \approx \frac{\ell d}{\ell^2 + d^2} = \frac{1}{2} \quad .$$

For $\Lambda = 0.5$, the series indicated in (6) converges to 4.0. The excitation then becomes

$$F = (6\,GMmd/R^3)\sin 2\,\Omega t \approx \frac{2GMm}{d^2}\sin 2\Omega t \quad . \qquad (7)$$

With the driving force given by (7), we may calculate the detector response from the second order equation for a damped harmonic oscillator:

$$\ddot{\alpha} + \frac{1}{2\tau}\dot{\alpha} + \omega_o^2\alpha = T/I \equiv \Gamma \sin 2\Omega t \qquad (8)$$

where

$\alpha \equiv$ displacement angle

$\omega_o \equiv$ resonant frequency

$\tau \equiv$ detector time constant

$T \equiv$ torque = Fd

$I \equiv$ moment of inertia = $2md^2$

$\Gamma \equiv$ gravitational gradient .

Evaluating the driving term,

$$\Gamma = \frac{GM}{d^3} , \qquad (9)$$

which has the units of sec^{-2} and which depends on the mass and size of the generator (which is the same size as the detector). These quantities are related, of course, so that we can reduce (9) to be a function only of density for a plate of fixed proportions.

Figure 5 demonstrates the meaning of the parameters in our quadrupole model of a flat plate. We fix the relative proportions to be such that the plates are ten times longer than wide, and one tenth as thick. The first relative dimension assures a dominant quadrupole moment, and also allows us to ignore the noninteracting circular mass distribution in the center. The second dimensional criterion (thickness) assures that the mass separation distance is negligible. Hence, the following relations hold for large dimension L:

$$M = \rho(L/2)\,(L/10)\,(L/100)$$
$$= \rho\, L^3/2000$$

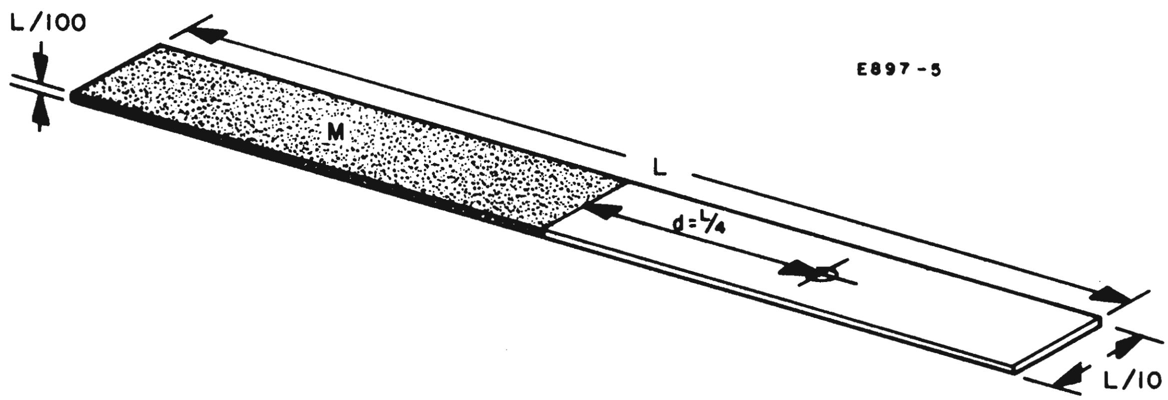

Fig. 5. Parameters of flat plate.

(where ρ is the specific density of the plate), since M corresponds to half the quadrupole moment. Also, $d = L/4$ since the average mass is approximately this distance from the axis of rotation.

Substituting these values into (9) yields

$$\Gamma = G\rho\ (L^3/2000)/(L^3/64)$$
$$= 0.03\ G\rho = 2\rho \times 10^{-9}\ \text{sec}^{-2} \qquad (10)$$

This indicates that the gravitational interaction is predominantly a function of the density of the material used in the generator. Because the material used must be nonconductive, nonmagnetic, and transparent, we shall be limited to densities of the various glasses. There are glasses such as lead oxide with a specific density of 9.5. However, the most dimensionally stable glass is fused quartz with 2.3. By proper choice of material we can hope to maintain dimensional stability while maximizing density. In this way, we can expect to obtain gradients on the order of $\Gamma = 10 \times 10^{-9}\ \text{sec}^{-2}$

The deflection due to a gravitational excitation is found from the solution to (8) at resonance (for zero initial conditions):

$$\alpha = (2\Gamma\tau/\omega_o)\ (1 - e^{-t/\tau})\ \sin\omega_o t \quad . \qquad (11)$$

For a fixed time constant τ, the amplitude for fully developed oscillation is inversely proportional to the natural frequency ω_o. It is therefore advantageous to use the lowest ω_o possible, from the standpoint of optical measurement of the deflection angle (Section B). The lower limit to ω_o is determined by the ability to support the resonant plate in a 1 g environment.

D. SIGNAL-TO-NOISE RATIO

The magnitude of angular deflection of the resonant plate due to gravitational coupling depends on the natural frequency and time constant τ (see eq. (11)). This amplitude must be large enough to dominate any nongravitational disturbance coupling by the factor of precision required in the experiment; in this case we consider a signal-to-noise ratio of 10^6.

In the previous dynamic gravitational experiments described in the Appendix, all sources of vibrational, acoustic, electromagnetic, etc., noise were negligible compared with the thermal noise in the device. This thermal noise is the limiting sensitivity in any instrument, because it is a thermodynamic molecular motion and depends only on temperature.

In order to determine the angular deflection of the gravitational detector due to the excitation of thermal noise, we will calculate the stored energy in a resonant harmonic oscillator. The energy is stored partly in the kinetic energy and partly in the potential energy of the spring:

$$E = KE + PE = \frac{1}{2} I \dot{\alpha}^2 + \frac{1}{2} \omega_o^2 \alpha^2 \quad . \tag{12}$$

The deflection α is obtained from the solution of (8) at steady state:

$$\alpha = \alpha_n \sin 2\Omega t \quad . \tag{13}$$

Substituting (13) into (12) yields (at $\omega_o = 2\Omega$):

$$E = \frac{1}{2} I(\omega^2 \alpha_n^2 \cos^2 \omega_o t) + \frac{1}{2} I \omega^2 \alpha_n^2 \sin^2 \omega_o t = \frac{1}{2} I \omega_o^2 \alpha_n^2 \quad . \tag{14}$$

The amplitude α_n of the oscillations, when the total energy in the vibrational mode is just the thermal energy (kT) required by the equipartition theorem, is

$$\alpha_n = \left(\frac{2\,kT}{I\,\omega_o^2}\right)^{1/2} = \left(\frac{2\,kT}{M}\right)^{1/2} \frac{1}{\omega_o d} = \left(\frac{1000\,kT}{\rho}\right)^{1/2} \frac{8}{\omega_o L^5} \quad . \tag{15}$$

Thus, the deflection due to thermal noise depends inversely on detector size and natural frequency.

To compare the thermal noise response with the gravitational response, we express the amplitude of (11) as

$$\alpha_g = 2\Gamma\, \tau/\omega_o \quad . \tag{16}$$

The signal-to-noise ratio is then obtained by dividing (16) by (15):

$$a_g/a_n = \frac{\Gamma\tau}{4}\left(\frac{\rho}{1000\ kT}\right)^{1/2} L^{5/2} \quad . \tag{17}$$

We see that a_g/a_n is proportional to the time constant and independent of natural frequency.

To obtain an accuracy of one part in 10^6, for example, we must have $a_g/a_n = 10^6$. Using this value, and a nominal dimension of $L = 50$ cm, we solve (17) for τ. The result is $\tau = 6 \times 10^4$ sec, or about half a day. This amount of time is a typical requirement for conducting a gravitational constant experiment,[7] and it is not unreasonable to maintain stability of the system parameters for that length of time.

When attempting to make ω_o as low as possible, the limit is determined by the ability of the sensitive torsion fiber to support the resonant plate in a 1 g environment. In the literature we find that natural periods of 1000 sec ($\omega_o \approx 0.01$ rad/sec) are common for torsional balances on the order of 100 g. A system with $L = 50$ cm weighs 625 g, which is not unreasonable for a low ω_o suspension.

Returning to (16) and using the values

$$\tau = 6 \times 10^4 \text{ sec}$$

$$\Gamma = 10 \text{ E.U.} = 10 \times 10^{-9} \text{sec}^{-2}$$

$$\omega_o = 0.1 \text{ rad/sec},$$

we calculate the gravitationally induced deflection of the resonator:

$$a_g = \frac{(2)\ (10 \times 10^{-9})\ (6 \times 10^4)}{(0.1)} \approx 0.01 \text{ rad} \quad ,$$

which is a fairly large angle. However, for the measurement to be precise to 10^{-6} implies a detection sensitivity of 10^{-8} rad. Using techniques described in Section B, we can expect to obtain this accuracy.

E. ERROR ANALYSIS

Equation (16) expresses the deflected angle as a function of the gravitational gradient, where $\Gamma = GM/d^3$ for the ideal mass quadrupole model. In the real case, the gravitational mass is not lumped at two discrete points, but instead is distributed over the extended dimensions of the plate. Thus, the values M and d represent the effective gravitational quadrupole moment of the rectangular mass distribution. These values must be calculated by techniques discussed in Section C (eq. (3)), following a precise measurement of the physical plate dimensions and total mass. However, it is of primary importance to facilitate a precise physical measurement. The mathematical conversion can always (in theory) be accomplished with any desired accuracy. Because of the flat-plate geometry, we can measure dimensions to greater accuracies than other geometries by using interferometer techniques. In this way, we can measure the dimensions and displacements to 5×10^{-7} cm.

To estimate the achievable accuracy of this flat plate dynamic Cavendish experiment, let us solve (16) for G (using (10)):

$$G = a_g d^3 \omega_o / 2\tau M \quad . \tag{18}$$

The percentage error in d will be $5 \times 10^{-7}/50 = 10^{-8}$, well within an accuracy of one part in 10^6. Also, since $M \approx 10^3$ g, the necessary precision requires only the measurement of milligrams, and this should not be limiting. Of course, the uniformity of density over the extended plate dimensions must also be accurate to one part in 10^6, but here again, because of the flat plate geometry and transparent materials, we can use optical techniques such as measuring the index of refraction.

We have already stipulated that for a_g to be measured to an accuracy of 10^{-6}, the amplitude must build up over many resonant oscillations to a value of 0.01 rad. (This required a system time constant of half a day.) Therefore, the accuracy of angular measurement should not be limiting. In addition, the system time constant τ is on the order of 10^5 sec, and thus needs to be accurately known only to 0.1 for the required experimental precision.

In addition to the above, this experimental method also requires the accurate determination of the resonant frequency ω_o. Even though the natural period $(2\pi/\omega_o)$ is very large (100 sec), a 10^{-6} requirement would mean that it must be accurate to 0.1 msec.

The separation distance is an implied parameter because, from (4), $R^2 = 2d^2 + h^2$. Thus, when $h << d$, as in the present case, h is only a second order consideration. For very closely spaced plates of several millimeters thickness, $h \sim 0.2$, whereas $d \sim 10$ cm.

The separation distance h is not as critical a parameter as may have been expected. This may be seen by the following:

$$R = \left[2d^2 + h^2\right]^{1/2} \approx \sqrt{2d}\ (1 + h^2/4d^2) \quad .$$

Evaluating, $h^2/4d^2 \approx 10^{-4}$, which need be known only to three significant places. To measure h to three significant figures requires an accuracy of only 10^{-3} cm.

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APPENDIX

Generation and Detection of Dynamic Gravitational-Gradient Fields

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Generation and Detection of Dynamic Gravitational-Gradient Fields*†

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(Received 5 August 1966)

We have constructed a generator of dynamic Newtonian gravitational-force-gradient fields and used it as a signal generator to calibrate the response of the gravitational-gradient detectors being developed in our research work on gravitational-mass sensors. The gravitational-gradient-field generator is a flat aluminum cylinder 14 cm in diameter, with four holes than can be filled with slugs of different density to create a rotating mass-quadrupole moment. The generator is mounted on an air-bearing-supported motor and rotated at a nominal speed of 44 rps (2640 rpm). Because of the bisymmetric mass distribution, the dynamic gravitational-gradient fields generated have a frequency of 88 Hz, or twice the rotation frequency. The detector is a 12-cm-diam cruciform-shaped structure which responds to 88 Hz gravitational-gradient forces. The small (10^{-11} cm) motions induced in the detector arms are sensed by piezoelectric strain transducers attached to the arms near the point of maximum strain. A simple vacuum system, an iron shield plate, and spring mounts suffice for acoustic and magnetic isolation, since most of the nongravitational noises were generated at 44 Hz, the rotation frequency, rather than at 88 Hz, the gravitational-gradient frequency. Data taken with four different mass distributions varying from 0 to 1000 g and separation distances varying from 4.8 to 12 cm agree well with the theory, indicating that only gravitational energy was being transmitted from the generator to the detector. The minimum dynamic gravitational-gradient field observed during this test was 6×10^{-9} $\sec^{-2}$ or 0.002 of the earth's gradient. The equivalent differential acceleration exerted on the sensor arms by this field was 3×10^{-11} g's.

INTRODUCTION

WE are engaged in a program to design, construct, and test a research model of a gravitational-mass sensor which can measure the mass of an object at a distance by using a rotating system of masses and springs (see Fig. 1) to detect the gravitational-force-gradient field of the object.[1,2] The ultimate objective of our work is to develop a small, rugged sensor to be used on spinning lunar orbiters to measure the mass distribution of the moon and on spinning deep space probes to measure the mass of the asteroids.

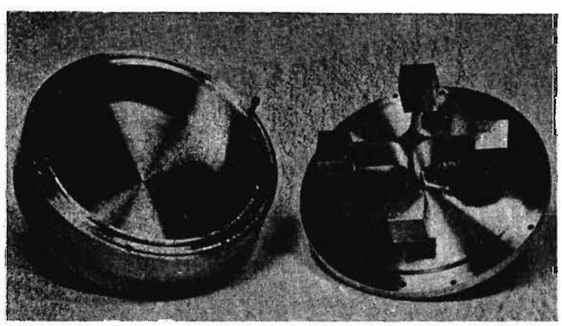

FIG. 1. Five-in.-diam cruciform gravitational-mass sensor.

Our primary goal in this research project is to develop methods of rotating the gravitational-mass-sensor structures without introducing large amounts of noise into the gravitational-gradient sensing mode, so that we can demonstrate the required degree of sensitivity in the laboratory without requiring flight tests to prove engineering feasibility. At present, we have demonstrated that we can measure accelerations down to 2×10^{-7} g's while operating in a 450-g rotational environment and a 1-g gravitational environment. The force level due to the earth's gravitational gradient is one order of magnitude below this. The noise problems are not fundamental and work is continuing on methods for lowering the noise level to the point where static gravitational gradients from laboratory masses can be seen.

A concurrent objective of our work is to learn enough about these structures to be able to predict their response to gravitational-gradient fields. The theoretical portion of this work is largely completed and was reported at the AIAA Second Annual Meeting.[3] In order to verity the equations experimentally and to develop a test system for calibrating the gravitational-gradient response of the various sensors, we have constructed a rotating generator of dynamic Newtonian gravitational-force-gradient fields and have measured the response of one of our sensors to these fields.[4] This work is similar to that of Weber *et al.* at the University of Maryland,[5] who utilized a vibrating rod to generate 1.6-kHz dynamic gravitational fields for calibration of a gravitation radiation detector.[6]

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‡ Presently on leave of absence on a Hughes Master of Science Fellowship at the University of California, Berkeley, Calif.
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DYNAMIC GRAVITATIONAL-GRADIENT-FIELD GENERATOR

The generator of the dynamic gravitational-gradient fields is shown in Fig. 2. The drive unit for the generator is an air-bearing support and drive which was originally designed to rotate a sensor structure. The bearing table supports an aluminum mass holder 14 cm in diameter with four holes, 5.0 cm in diameter and 3.5 cm deep, on a radius of 4.0 cm. Opposite pairs of holes can be filled with either aluminum, brass, or tungsten slugs which slip fit into the holes. The various pairs of mass slugs were trimmed so that static and dynamic balance of the generator was achieved even though the mass holder has a mass-quadrupole moment. When balanced, the motor–generator combination is silent under all combinations of speed and mass-quadrupole loading, except for a slight, high-frequency hiss of the support air passing through the bearing. The motor can be operated in either a synchronous drive mode or a phase-locked asynchronous mode. The readout of the generator rotation speed and phase is obtained through a photoelectric pickoff which detects paint marks on the rotor. This photoelectric signal is used as the reference signal for a lock-in amplifier, and in the asynchronous mode can also be used to supply pulses for the asynchronous drive controller.

The masses of the various slugs used are

Tungsten	1212.0 g,
Brass	606.0 g,
Aluminum	200.0 g.

If four aluminum slugs are used, the generator has no mass-quadrupole moment. The maximum mass-quadrupole moment of 3.8×10^4 g·cm² is obtained when two tungsten slugs are used and the other two holes are left empty. When the opposing pair of holes is filled, the effective mass is just the mass difference. The various combinations possible with our present setup are listed below.

Holes 1 and 3	*Holes 2 and 4*	*Effective Mass, g*
Tungsten	Empty	1212.0
Tungsten	Aluminum	1012.0
Tungsten	Brass	606.0
Brass	Empty	606.0
Brass	Aluminum	406.0
Aluminum	Empty	200.0
Aluminum	Aluminum	0.0

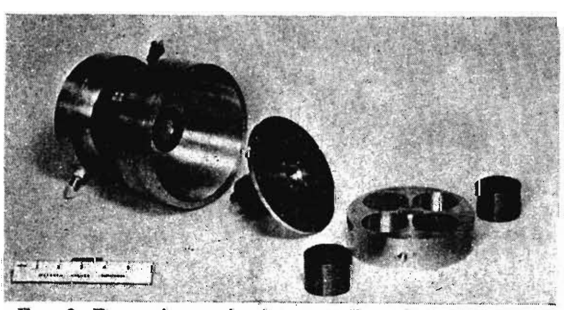

FIG. 2. Dynamic gravitational-gradient-field generator.

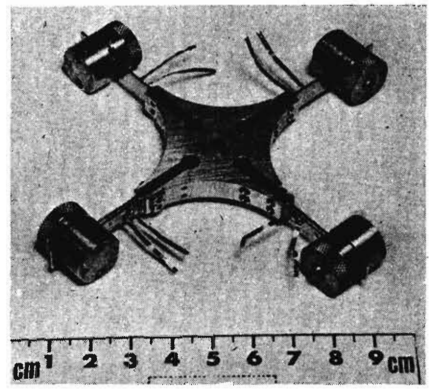

FIG. 3. Adjustable sensor.

The generator rotates at a nominal speed of 44 rps (2640 rpm); because of the bilateral or tensorial character of the mass-quadrupole generators, the ac gravitational-gradient fields generated are at 88 Hz, or *twice* the rotation frequency. (See Appendix.)

DETECTOR

The detector used in this first test was one of our adjustable sensors (see Fig. 3). The sensing masses of the detector are 20-g brass weights attached to the sensor arms by a screw-clamp arrangement. The weights have an eccentric cam arrangement which allows for small position adjustments on the arms. The arms are cantilever beams of aluminum with a 0.125-in.-thick base where they fasten to the hub and an outer bending portion 0.030 in. thick and about 0.70 in. long. The aluminum hub is designed to clamp the arms rigidly for good cross coupling and yet allow the arm-mass assembly to be moved in and out for mass balance of the final assembly.

The detector has a resonant frequency of 88.45 Hz in the dual tuning fork or gravity-gradient sensing mode (see Fig. 7 in the Appendix), a Q of 120 and an arm length of $l=5.0$ cm. Under the influence of a gravitational gradient of $\Gamma\ \sin2\Omega t$, the arms respond with a vibrational amplitude of [see Eq. (A21) of the Appendix]

$$\Delta=[Ql/(2\Omega)^2]\Gamma\cos2\Omega t=1.95\times10^{-3}\ \text{cm/sec}^2\ \Gamma\cos2\Omega t, \quad (1)$$

where $2\Omega=2\pi\times88.45$ rad/sec.

The readout of the detector vibrations is accomplished by sensing the dynamic strains of the detector arms with barium titanate strain transducers. (Gulton Ind. type SC-2). A pair of transducers were reversed from the arrangement shown in Fig. 3 so that opposing pairs of transducers would produce a differential output voltage which could be fed into the differential input of a Princeton Applied Research HR-8 lock-in amplifier.

The dynamic strain in the arms due to their deflection is a strong function of the details of the design of the detector arms, and is difficult to calculate accurately because of the complex mechanical structure used. The relationship predicted in Ref. 3 is

$$\epsilon=[(b+L)c/(\tfrac{1}{3}L^3+bL^2+b^2L)]\Delta=0.026\ \text{cm}^{-1}\Delta, \quad (2)$$

where $b=0.3$ cm is the radius of the end mass, $L=1.8$ cm is the length of the arm, and $c=0.038$ cm is the half-thickness of the arm.

The barium titanate strain transducers extend over a considerable portion of the arm; therefore, they measure an averaged value of the strain, which is a maximum at the hub and zero at the end. This average measured strain was estimated as

$$\epsilon_t=0.6\epsilon=0.016\ \text{cm}^{-1}\Delta. \quad (3)$$

The transducers used on the detector had been calibrated on a test setup which compared them with a resistive strain gauge using pure longitudinal strains at 1600 Hz. The transducer factor obtained under these conditions was about $\sigma=0.7\times10^5$ V/unit strain. Thus the voltage output from this sensor should be approximately

$$V=\sigma\epsilon_t=1.1\times10^3\ \text{V/cm}\ \Delta=2.2\ \text{V}\cdot\text{sec}^2\ \Gamma\cos2\Omega t. \quad (4)$$

NONGRAVITATIONAL COUPLING

It is obvious that in order to detect the very weak dynamic gravitational forces being generated by the rotating mass quadrupole, the generator and detector must be well shielded from each other to prevent acoustic and electromagnetic coupling. The detector is highly sensitive to acoustic noise with a frequency component at its resonance frequency, but experience has shown that the acoustic noise can be eliminated by placing the detector in a vacuum chamber at a few mTorr.

Although an ideal detector is theoretically insensitive to vibrations of the mounting structure,[3] in practice a small amount of the vibrations in the mount leak into the gradient-sensing mode. Because of this, an effort must be made to keep the detector-mount vibrations at a low level. This was accomplished by suspending the detector in the chamber with a spring, and the chamber from the ceiling by another spring. The generator was isolated from the workbench by compression springs, and the iron-shield plate was vibrationally isolated from both the generator and detector by its own support springs (see Fig. 4).

Electromagnetic coupling can occur in two ways: (1) by direct interaction of the rotating magnetic fields of the motor with the arms of the detector; (2) by stray electromagnetic voltages or currents entering the detector output leads or the preamplifier. The electromagnetic coupling into the output electronics is easily checked, since it is independent of the resonant response of the detector and was found to be unobservable even in the single-ended mode of operation, although all data were taken with a differential input to insure that pickup was not a problem.

Direct coupling of the rotating magnetic fields around the generator motor into the detector was found to be a major problem. At first it was not well understood, since the detector arms were of aluminum and the detector masses of brass. This interaction was originally eliminated by using a phase-locked asynchronous drive. In this mode of operation of the generator, the generator motor is driven by currents at some higher frequency, typically 200 Hz, so they do not excite the detector resonant mode. The amplitude of the drive voltage is controlled by a servo loop so that the rotor remains at a constant speed of 44 rps. The servo loop is so tight that both the frequency and the phase of the rotor are held tightly to the phase of a reference signal from a precise oscillator (General Radio frequency synthesizer). It was later discovered that the detector had been assembled with stainless steel screws; when they were replaced by brass screws, the magnetic coupling was eliminated and it was possible to take good data using synchronous drive on the generator.

One important factor aided greatly in the problem of eliminating the nongravitational coupling between the generator and the detector. Because of the double mass in the mass quadrupole, the generator is rotated at *half* of the detector frequency. Therefore, predominantly all of the acoustic and electromagnetic energy produced by the generator is at a frequency which is outside the detector-response frequency; only that small portion of the energy which is harmonically generated at twice the rotation frequency must be shielded against.

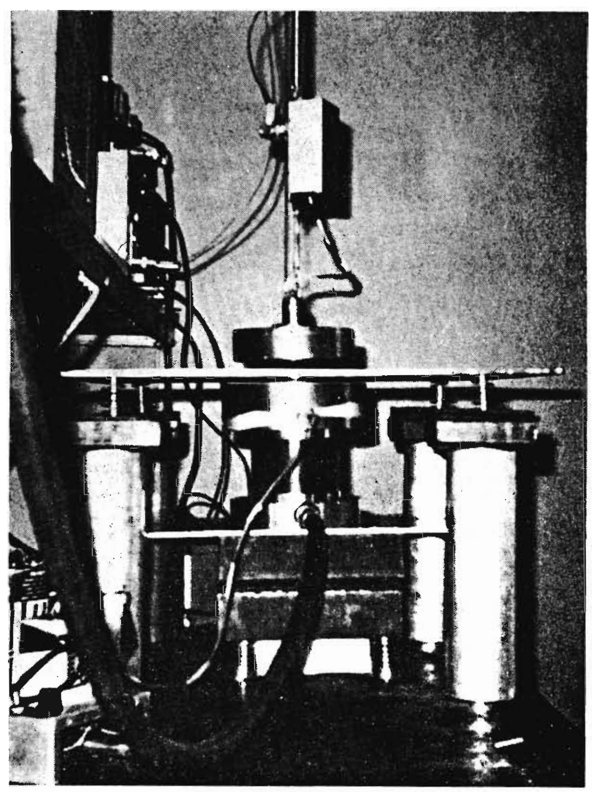

FIG. 4. Relative position of generator and detector.

The generator was designed specifically for the problem of determining the nongravitational-coupling effects. If four aluminum slugs are put in the mass holder, the generator has no time-varying mass-quadrupole moment and therefore no dynamic gravitational-gradient field; however, it still retains all of its electromagnetic and acoustic properties. A test run was made at 5 cm separation distance using the four aluminum slugs. The generator speed was varied from 43 to 45 rps, so that the detector-mode frequency of 88 Hz was not missed. The detector output remained at 0±4 nV. The rotor was then deliberately unbalanced so that the acoustic output was noticeably increased and the test was rerun, with the same results. These experiments demonstrated that the response of the detector structure and sensor electronics to nongravitational forces arising from all sources, including the generator, was less than 4 nV.

DETECTOR CALIBRATION

After the test for nongravitational coupling, two of the aluminum slugs were replaced with tungsten slugs, resulting in a mass difference of 1012 g. The rotor was rebalanced and the generator and detector were placed 5 cm apart. The theoretical calculations presented in the Appendix indicate that at this distance, and with this size detector, a 1012-g effective mass should produce an equivalent gravitational-force gradient of

$$\Gamma \sin 2\Omega t = 1.25\times 10^{-7} \sin 2\Omega t \ \mathrm{sec}^{-2}. \qquad (5)$$

The dynamic gradient has an amplitude of about 0.04 of the earth's static gravitational-force gradient.

From the theory of operation of the sensors,[3] this gradient should cause the gravitational-gradient sensing mode of the detector to oscillate with an amplitude of [see Eq. (1)]

$$\Delta = 2.5\times 10^{-10} \cos 2\Omega t \ \mathrm{cm}. \qquad (6)$$

Although the amplitude of these motions is extremely small, of the order of 0.01 of the diameter of an atom, they are easily measured if piezoelectric strain transducers are used. Similar sensing techniques used on the gravitational radiation detectors at the University of Maryland[5,6] have measured motions down to 10^{-14} cm.

The motion induced in the detector causes an average strain in the arms of [see Eq. (3)]

$$\epsilon_t = 3.9\times 10^{-12} \cos 2\Omega t. \qquad (7)$$

If we assume that the transducer calibration is $\sigma = 0.7\times 10^{5}$ V/strain, the predicted output of these sensors under excitation by a generator with a 1-kg mass difference at a 5-cm separation distance would be [see Eq. (4)]

$$V = 2.2\ \mathrm{V/sec^2}\ \Gamma \cos 2\Omega t$$

$$= 270 \cos 2\Omega t \ \mathrm{nV\ (predicted)}, \qquad (8)$$

or an rms voltage of 190 nV.

When the test was run, the actual measured output voltage of one arm of the sensor under these conditions was 97±3 nV (rms). This is much larger than the output-voltage fluctuations of ±4 nV under the control conditions using the four aluminum masses, and is almost exactly half the predicted output. The reason for this lower output is not known. It is assumed that it is a result of the difficulty in obtaining an accurate calibration of the strain transducers, or in calculating the strain from the deflection Δ. Further experiments are in progress to resolve the question. The gravity-gradient input to detector-voltage output relationship for the adjustable detector obtained from this calibration is

$$V = 1.1\ \mathrm{V/sec^2}\ \Gamma \cos 2\Omega t. \qquad (9)$$

VERIFICATION OF GRAVITATIONAL COUPLING

Although the control experiments with the four aluminum slugs and the balanced and unbalanced rotor indicated that the nongravitational coupling was negligible, it was still possible that the replacement of the aluminum slugs with the tungsten slugs could change the magnetic moment or balance of the generator and cause nongravitational coupling. In order to further insure that the voltage output seen was caused only by gravitational-gradient coupling, a run of data was taken at various separation distances and with various mass-quadrupole moments. (One of the aluminum slugs froze in its hole in the generator during the preliminary work so it was possible to try only four different mass-quadrupole arrangements.)

At the start of the experiment, the phase of the lock-in detector was adjusted to give a maximum output with the tungsten slugs and was not adjusted or peaked during the remainder of the data run. The quadrature voltage was monitored periodically to insure the detection of any phase shift in the signal induced by any variation in the relative strength of the gravitational coupling and any synchronous nongravitational coupling. No quadrature component was detected during the data runs.

With the tungsten slugs in the generator, a set of data was taken while the separation distance was varied from 4.8 cm to 12 cm. The generator was then stopped and the tungsten slugs replaced with brass slugs, resulting in an effective mass difference of 406.0 g. Without adjustment to the sensor electronics, a second set of data was taken from 4.8 to 10 cm. The generator was again stopped and the brass slugs removed, leaving a void or relative mass difference of −200 g. The phase knob of the lock-in detector was switched exactly 180° to account for the effective negative mass, or 180° signal-phase difference, and the third set of data taken from 4.8 cm to 8 cm. When aluminum slugs were placed in all four holes, the output was 0±3 nV. The data are plotted in Fig. 5.

Curves of detector output versus separation distance were then calculated and plotted in Fig. 5 for various

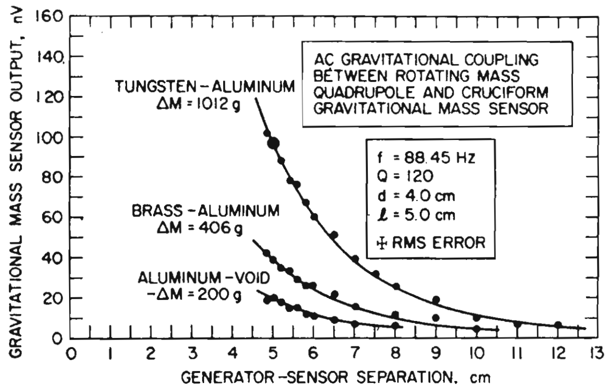

FIG. 5. Dynamic gravitational coupling between rotating mass quadrupole and cruciform gravitational-mass sensor.

mass-quadrupole moments using the theoretical Eqs. (A-15) and (A-16) derived in the Appendix. For conversion of the calculated equivalent gravitational-gradient field to sensor-voltage output, we used the calibration point at 5 cm and 97 nV (larger data point). The two lower curves are the upper curve multiplied by 0.4 and 0.2, respectively.

The excellent agreement of the data with the theoretical predictions in amplitude and phase for various conditions of mass-quadrupole moment and separation distance indicates that only gravitational energy was being transmitted from the generator to the detector. The minimum dynamic gravitational-gradient field observed during this test was about 6×10^{-9} sec^{-2} (6 Eötvös units) or 0.002 of the earth's gradient. The effective differential acceleration on the 5-cm-long detector arms due to this field was

$$a=\Gamma l=3\times10^{-8}\ \text{cm/sec}^2=3\times10^{-11}\ g\text{'s}, \qquad (10)$$

and the effective force level on the 20-g detector masses was

$$F=ma=6\times10^{-7}\ \text{dyn}. \qquad (11)$$

SUMMARY

We have constructed a generator of 88-Hz gravitational-gradient fields and used the fields to calibrate the response of a dynamic gravitational-gradient sensor. The test involved the transmission of gravitational energy over distances up to 12 cm by means of dynamic Newtonian gravitational-gradient fields.

APPENDIX: GRAVITATIONAL INTERACTIONS BETWEEN A CRUCIFORM DETECTOR AND A ROTATING-MASS QUADRUPOLE

The model which we will use to calculate the gravitational interaction between a rotating-mass quadrupole and a resonant cruciform gravitational-mass sensor is shown in Fig. 6. The generator is assumed to be two spherical masses of mass M separated by a distance $2d$ and rotated about their center of mass at a constant angular frequency $\dot{\theta}=\Omega$. The detector is assumed to be four spherical masses of mass m on orthogonally disposed massless arms of length l. The sensor is supported from above so that its center of mass is at a height h directly above the center of mass of the generator. The particular mode of the sensor used to sense the gravitational-gradient forces is the dual tuning-fork mode (see Fig. 7). It was shown in previous analyses[3] that this mode does not respond to vibrational forces at the mount nor to the direct gravitational force field, but only to the gradient of the gravitational force field.

The forces on the sensor resulting from the gravitational interaction between the rotating masses M_c and the sensor masses m_i typically consist of

$$F_{ic}=-GMm/R_{ic}^2,\quad i=1\text{ to }4,\ c=a,\ b,\ m_i=m,\ M_c=M, \qquad \text{(A1)}$$

where

$$R_{ic}^2=h^2+r_{ic}^2, \qquad \text{(A2)}$$

and

$$r_{3a}^2=r_{1a}^2=l^2+d^2-2ld\cos\theta,$$

$$r_{3b}^2=r_{1b}^2=l^2+d^2+2ld\cos\theta,$$

$$r_{4a}^2=r_{2a}^2=l^2+d^2-2ld\sin\theta,$$

$$r_{4b}^2=r_{2b}^2=l^2+d^2+2ld\sin\theta. \qquad \text{(A3)}$$

However, the components of the forces which drive the sensing mode of the detector are the tangential

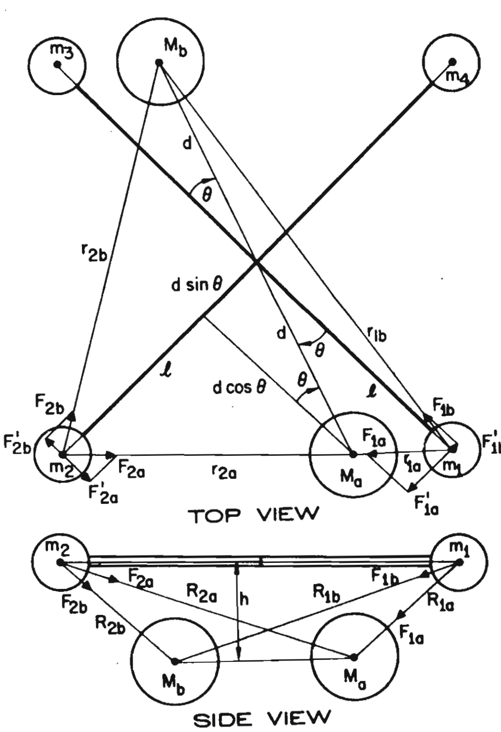

FIG. 6. Model for gravitational-interaction calculation.

components of the forces F_{ic}

$$\begin{aligned}
F_{3a}'&=F_{1a}'=(GMm/R_{1a}^3)\,d\sin\theta,\\
F_{3b}'&=F_{1b}'=-(GMm/R_{1b}^3)\,d\sin\theta,\\
F_{4a}'&=F_{2a}'=-(GMm/R_{2a}^3)\,d\cos\theta,\\
F_{4b}'&=F_{2b}'=(GMm/R_{2b}^3)\,d\cos\theta,
\end{aligned}\tag{A4}$$

where the sense of the forces is taken to be positive in the clockwise direction.

The resultant force F_i' on each of the arms due to the forces F_{ic}' is given by

$$\begin{aligned}
F_3'&=F_1'=F_{1a}'+F_{1b}'\\
&=GMmd[(1/R_{1a}^3)-(1/R_{1b}^3)]\sin\theta,\\
F_4'&=F_2'=F_{2a}'+F_{2b}'\\
&=-GMmd[(1/R_{2a}^3)-(1/R_{2b}^3)]\cos\theta.
\end{aligned}\tag{A5}$$

The response of the detector arms to these resultant forces has been presented in a previous work by Bell, Forward, and Morris.[3] Taking a simplified version of their Eqs. (38) through (41), we obtain

$$m\ddot{\Delta}_i+d\dot{\Delta}_i+k\Delta_i=F_i',\qquad i=1\text{–}4,\tag{A6}$$

where Δ_i is the deflection of the ith arm, k is the spring constant, and d is the damping.

These four equations (A6) describe the individual motions of the four arms; however, the vibrations of interest are the motions of the gravity-gradient sensing mode (see Fig. 7). The equation for this mode is obtained by the following linear combination of the individual arm motions[7]

$$\Delta_g=\tfrac{1}{2}(\Delta_1-\Delta_2+\Delta_3-\Delta_4),\tag{A7}$$

where the normalization factor of $\frac{1}{2}$ is used so that the detector energy expressed in terms of the four arm amplitudes is equivalent to the detector energy expressed in terms of the four vibrational mode amplitudes (gravity gradient, torsional, and two translational).[7]

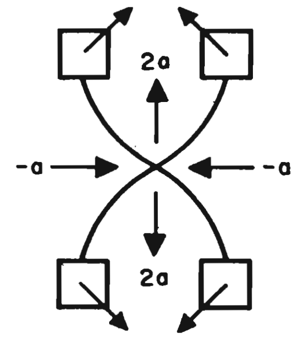

FIG. 7. Gravity-gradient sensing mode.

If we add the equations for the four arm motions [Eqs. (A6)] in this manner, we obtain the equation of motion for the gravity-gradient sensing mode

$$m\ddot{\Delta}_g+d\dot{\Delta}_g+k\Delta_g=\tfrac{1}{2}(F_1'-F_2'+F_3'-F_4')=F_g,\tag{A8}$$

where

$$F_g=GMmd\{[(1/R_{1a}^3)-(1/R_{1b}^3)]\sin\theta+[(1/R_{2a}^3)-(1/R_{2b}^3)]\cos\theta\}.\tag{A9}$$

Because R_{ic} is a function of the angle θ, the resultant force F_g has a complex behavior with the angle of rotation. To calculate the components of the resultant force as a function of frequency, we will expand the terms in R_{ic}^{-3}. Letting

$$R^2=l^2+d^2+h^2,\tag{A10}$$

we can write the resultant force F_g as

$$F_g=GMmd\{[(R^2-2ld\cos\theta)^{-3/2}-(R^2+2ld\cos\theta)^{-3/2}]\sin\theta+[(R^2-2ld\sin\theta)^{-3/2}-(R^2+2ld\sin\theta)^{-3/2}]\cos\theta\}.\tag{A11}$$

Bringing R^2 out from the denominator and letting $\Lambda=(ld/R^2)$, we obtain

$$F_g=(GMmd/R^3)\{[(1-2\Lambda\cos\theta)^{-3/2}-(1+2\Lambda\cos\theta)^{-3/2}]\sin\theta+[(1-2\Lambda\sin\theta)^{-3/2}-(1+2\Lambda\sin\theta)^{-3/2}]\cos\theta\}.\tag{A12}$$

We now expand each term using the binomial expansion theorem; however, because the expansion parameter Λ can be as high as $\frac{1}{3}$ when the generator and detector are separated by 4.5 cm, it is necessary to take the expansion out to the seventh order.

$$\begin{aligned}
F_g=(GMmd/R^3)\{&[6\Lambda\cos\theta+35\Lambda^3\cos^3\theta+\tfrac{693}{4}\Lambda^5\cos^5\theta+\tfrac{6435}{8}\Lambda^7\cos^7\theta]\sin\theta\\
&+[6\Lambda\sin\theta+35\Lambda^3\sin^3\theta+\tfrac{693}{4}\Lambda^5\sin^5\theta+\tfrac{6435}{8}\Lambda^7\sin^7\theta]\cos\theta\}.
\end{aligned}\tag{A13}$$

(The even order terms drop out because of the symmetry.) If we rearrange the above equation and use the trigonometric identities

$$\begin{aligned}
2\sin\theta\cos\theta&=\sin2\theta,\\
2(\cos^3\theta\sin\theta+\sin^3\theta\cos\theta)&=\sin2\theta,\\
16(\cos^5\theta\sin\theta+\sin^5\theta\cos\theta)&=5\sin2\theta+\sin6\theta,\\
32(\cos^7\theta\sin\theta+\sin^7\theta\cos\theta)&=7\sin2\theta+3\sin6\theta,
\end{aligned}\tag{A14}$$

we can obtain the expression

$$F_g=(6GMmld^2/R^5)\{(1+\tfrac{35}{12}\Lambda^2+\tfrac{1155}{128}\Lambda^4+\tfrac{15015}{512}\Lambda^6)\sin2\Omega t+(\tfrac{231}{128}\Lambda^4+\tfrac{6435}{512}\Lambda^6)\sin6\Omega t\},\tag{A15}$$

[7] C. C. Bell, J. R. Morris, J. M. Richardson, and R. L. Forward, "Vibrational Mode Behavior of Rotating Gravitational Gradient Sensors" (to be published).

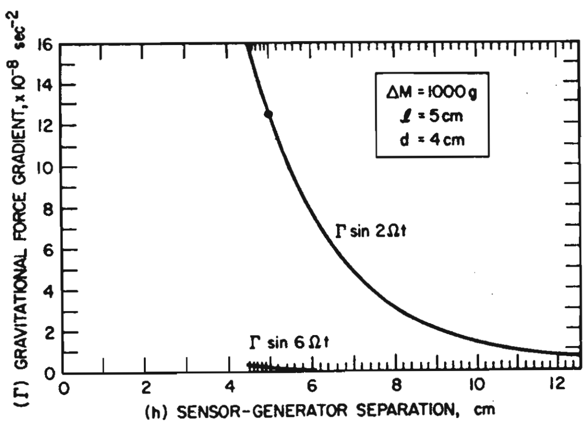

Fig. 8. Calculated equivalent gravitational-force gradient.

where we brought out a factor of $6\Lambda = 6ld/R^2$ from behind the brackets. This expression shows that the interaction force between the generator and detector is complicated at close distances of separation and depends upon the sizes of the generator and detector, as well as the separation distance. This expression also shows that in addition to responding to the gravitational-force gradient or the second-order gradient of the potential at 2Ω, the detector will also respond to the sixth-order gradient of the potential at 6Ω. Because of the symmetry of the generator–detector combination, the intermediate higher-order gradients are not observable.

In order to relate the equation for the effective force on the gravitational-gradient sensing mode of the detector [Eq. (A15)] to the previous work, we define an equivalent gravitational-force gradient by the relation

$$\Gamma = F_g/2ml, \qquad \text{(A16)}$$

where m is the effective mass and $2l$ is the effective length of the gravitational-gradient sensing mode.

The effective gravitational-force gradient [Eq. (A16)] was computer calculated for various values of the separation distance h, and the results for the amplitudes of the two frequency components are plotted in Fig. 8. For this curve it was assumed that the detector had an effective radius of 5 cm, and the generator consisted of two 1-kg masses on a radius of 4 cm. At the nominal separation distance of 5 cm, the effective gravitational-force gradient resulting from the generator is 1.24×10^{-7} sec^{-2}. This is about 0.04 of the earth's gradient. These two relatively small masses have a relatively large gradient because we are able to bring the center of mass of the detector very close to the center of mass of the generator.

At distances greater than 12 cm, the only important term in Eq. (A16) is the first, and the effective gravitational gradient is given by the formula

$$\Gamma = 3GMd^2 \sin 2\Omega t/(h^2+d^2+l^2)^{5/2} \approx (3GMd^2/h^5) \sin 2\Omega t. \qquad \text{(A17)}$$

The gradient is falling off as d^2/h^5 rather than as $1/h^3$ because the detector is only sensitive to the dynamic gradient being generated by the rotating mass-quadrupole moment of the generator and is not sensitive to the static gradient of its monopole moment which does fall off as $1/h^3$.

If we choose the rotation speed Ω of the generator so that the detector senses the gravitational-force gradient fields being generated at twice the rotation speed

$$k/m = (2\Omega)^2, \qquad \text{(A18)}$$

then the gravitational forces at 2Ω are seen to be driving terms in the equation of motion of the vibrational mode [Eq. (A8)]:

$$\ddot{\Delta}_g + (d/m)\dot{\Delta}_g + (2\Omega)^2\Delta_g = 2\Gamma l \sin 2\Omega t. \qquad \text{(A19)}$$

The solution to this equation is well known as

$$\Delta_g = -2\Gamma l[Q/(2\Omega)^2] \cos 2\Omega t, \qquad \text{(A20)}$$

where Δ_g is the amplitude of the vibrational mode and $Q = 2\Omega m/d$ is the quality factor of the resonance.

In practice we do not measure the mode amplitude directly, but instead measure the amplitude of one of the arms

$$\Delta_1 = \tfrac{1}{2}\Delta_g = -\Gamma l[Q/(2\Omega)^2] \cos 2\Omega t. \qquad \text{(A21)}$$