A SPIN-3/2 THEORY OF GRAVITATION [1]

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ABSTRACT

Stimulated by ideas occuring in Supergravity we develop a gauge theory of gravity based on a spin-3/2 Majorana field. Our theory has no metric or vierbein as an elementary field. Classically the theory is in complete agreement with Einstein's metric formulation, but quantum mechanically it differs from ordinary formulations, including supergravity, on the fundamental nature of gravitation. In our approach gravitation arises from a collective effect due to spin-3/2 gravitinos.

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Supergravity[1] is a theory of gravitation and of a Majorana spin-3/2 field, invariant under a local supersymmetry transformation. Inspired by this idea we envisaged the possibility of constructing a theory of gravitation in which the metric tensor is obtained as a composite field in terms of the elementary spin-3/2 field alone.[2]

In this essay we propose a generally covariant and locally Lorentz invariant theory of a Mojorana spin-3/2 field ψ_μ^α. It differs from supergravity by the absence of an elementary vierbein field h_μ^i. Nevertheless we show that, through an appropriate ansatz, a classical equation can be obtained for a "collective" metric tensor, which exactly coincides with Einstein's equation in general relativity. Thus we interpret Einstein's theory of gravity as a special classical solution of our theory, similar in spirit to e. g. soliton or monopole solutions of classical field equations. This classical solution provides a curved background on which the Fermi field is quantized by taking:

$$\psi_\mu^\alpha = (\psi_\mu^\alpha)_{classical} + (\psi_\mu^\alpha)_{quantum} \tag{1}$$

This approach is analogous to quantization in the presence of a background field, often used in general relativity.

In the language of the geometry of fibre bundles we consider a principal bundle with a four dimensional space-time manifold M_4 as a base and the Lorentz group $G = SL(2C)$ as the structure group. A connection is introduced in terms of an equivariant horizontal form A with values in the Lie algebra of G, whose components (in a coordinate basis) are the gauge potentials A_μ^{ij}. We also consider an associated bundle with the fibre ψ being a real spinor-valued horizontal one-form belonging to the $(1/2, 0) \oplus (0, 1/2)$ representation of $SL(2C)$. The induced connection is given by $A = A^{ij}\sigma_{ij}$ where $\sigma_{ij} = 1/2(\gamma_i\gamma_j - \gamma_j\gamma_i)$ are real Dirac matrices (Majorana representation), generators of Lorentz transformations. We define the covariant derivatives D_μ in a coordinate basis $\{\partial_\mu\}$ by:

$$D_\mu = \partial_\mu + A_\mu^{ij}\sigma_{ij} \tag{2}$$

The curvature R is a horizontal two-form with values in the Lie algebra of G whose components in a coordinate basis are given by:

$$[D_\mu , D_\nu] = - R_{\mu\nu}^{ij}\ \sigma_{ij} \tag{3}$$

or

$$R_{\mu\nu} = \partial_\nu A_\mu - \partial_\nu A_\mu + [A_\nu, A_\mu] \tag{4}$$

We take the action to be given by:

$$S = \frac{1}{K} \int (D\psi)^{\alpha} \wedge (D\psi)^{\beta} \, (\gamma_5)_{\alpha\beta} \tag{5}$$

where K is a constant, the pseudoscalar Dirac matrix $(\gamma_5)_{\alpha\beta}$ is real and antisymmetric, and we have used the notation: $(D\psi)^{\alpha} = (D_{\mu}\psi)^{\alpha} \underline{dx}^{\mu} \wedge \underline{dx}^{\nu}$. The metric that raises and lowers spinor indices is the charge conjugation matrix $C = \gamma_0$ in the Majorana representation. Notice that the action (5) is non-vanishing only if the components of ψ are non-commuting.

It is instructive at this point to compare this action with that for general relativity in the vierbein formalism. Introducing a vector-valued horizontal one-form h (the vierbein) transforming as the (1/2, 1/2) representation of SL (2C) one can write Einstein's action in the form:

$$S_{\text{Einstein}} = \frac{1}{K} \int h^{i} \wedge R^{jk} \wedge h^{l} \quad \epsilon_{ijkl} \tag{6}$$

On the other hand upon integration by parts (5) becomes:

$$S = \frac{1}{K} \int \psi^{\alpha} \wedge R^{jk} \wedge \psi^{\beta} \quad \sigma^{il}_{\alpha\beta} \; \epsilon_{ijkl} \tag{7}$$

in close analogy with (6). Carrying the analogy further it will be natural to introduce a metric tensor as a composite field defined (up to a constant factor) by:

$$g_{\mu\nu} = \psi_\mu^\alpha \, C_{\alpha\beta} \, \psi_\nu^\beta \, . \tag{8}$$

We also notice that the two-form $(D\psi)^\alpha \equiv R^\alpha$, which appears in the action (5), can be identified with the spinor component of the curvature tensor $R = \{R^{ij}, R^\alpha\}$ for a bundle space over M_4 with structural supergroup OSp(1,2c). This suggests the possible existance of a supersymmetry invariance. Indeed we find that the action (5) is invariant under the following transformation:[4]

$$\delta\psi_\mu^\alpha = (D_\mu \epsilon)^\alpha = \partial_\mu \epsilon^\alpha + (A_\mu \epsilon)^\alpha \tag{9}$$

$$(\delta A_\mu \psi_\nu - \delta A_\nu \psi_\mu)^\alpha = -(R_{\mu\nu} \epsilon)^\alpha \tag{10}$$

This last equation may be only formal, since it is not clear that it can be solved for δA_μ. Nevertheless a conserved current associated with the corresponding global symmetry can be derived via Noether's theorem:

$$J_\alpha^\mu = \varepsilon^{\mu\nu\lambda\sigma} (\gamma_5 A_\nu D_\lambda \psi_\sigma)_\alpha \tag{11}$$

The equations of motion following from the action principle $\delta S = 0$ are:

$$\psi \gamma_5 \sigma_{ij} \wedge D\psi = 0 \tag{12}$$

$$R \wedge \psi = 0 \tag{13}$$

Using Bianchi's identity it is easy to verify that the last equation yields $\partial_\mu J^\mu_\alpha = 0$.

We shall now look for a classical solution to these equations by making the ansatz that

$$A^{ij}_\mu = a^{ij}_\mu \quad , \quad \psi^\alpha_\mu = h^i_\mu (\gamma_i)^\alpha{}_\beta \, \theta^\beta \tag{14}$$

where a^{ij}_μ and h^i_μ are classical functions and θ is a constant spinor. It will be shown later that θ is related to the supersymmetry charge associated with the conserved spinor current (11). Strictly speaking, since ψ is a non-commuting spinor, there does not exist a completely classical solution. What is meant by the ansatz (14) is that it describes states $|B\rangle$ and $|B\alpha\rangle = \theta_\alpha |B\rangle$

such that:

$$\langle B| A_\mu^{ij} |B\rangle = a_\mu^{ij}$$
$$\langle B| \psi_\mu^\alpha |B_\beta\rangle = h_\mu^i (\gamma_i)^\alpha{}_\beta \tag{15}$$

The fields quantized about the state $|B\rangle$ will then be given by

$$A_\mu = a_\mu + A'_\mu \quad , \quad \psi_\mu = h_\mu^i \gamma_i \theta + \psi'_\mu \tag{16}$$

Substituting (14) in the equations of motion (12,13) we obtain the following equations for h_μ^i and a_μ^{ij}:

$$\varepsilon^{\mu\nu\lambda\sigma} (D_\lambda h_\sigma)^i = 0 \tag{17}$$

$$\varepsilon^{\mu\nu\lambda\sigma} R_{\mu\nu}^{ij} h_\lambda^k [\varepsilon_{ijk\ell} \gamma_5 \gamma^\ell + \eta_{jk}\gamma_i - \eta_{ik}\gamma_j] = 0 \tag{18}$$

where

$$(D_\lambda h_\sigma)^i = \partial_\lambda h_\sigma^i + A_\lambda^{ij} h_\sigma^k \eta_{jk} \tag{19}$$

The second term in eq. (18) vanishes as a result of applying D_ν to eq. (17). Then (18) reduces to:

$$\varepsilon^{\mu\nu\lambda\sigma} R_{\mu\nu}^{ij} h_\lambda^k \varepsilon_{ijkl} = 0 \tag{20}$$

Eqs. (17) and (20) are identical to Einstein's equations in the absence of sources, in the vierbein formalism. Eq. (17) is the condition of zero torsion and completely determines a_μ^{ij} in terms of h_μ^i. After standard manipulations eq. (20) can be brought to the more familiar form $R_\mu{}^\nu = 0$ where $R_\mu{}^\nu = R_{\mu\lambda}{}^{ij} h_i^\lambda h_j^\nu$ is the Ricci tensor. The metric tensor defined by (8) now becomes $g_{\mu\nu} = h_\mu^i h_\nu^j \eta_{ij} \theta C \theta = (g_{\mu\nu})_{\text{classical}} \theta C \theta$.

There remains to understand the meaning of θ. We define the total charge S_α associated with the fermionic conserved current by

$$S_\alpha = \int_\Sigma J_\alpha^\mu \, d\sigma_\mu \tag{21}$$

where Σ is a spacelike surface with respect to the classical metric. Using the equations of motion, J_α^μ can be written as a total divergence $J_\alpha^\mu = \varepsilon^{\mu\nu\lambda\sigma} \gamma_5 \partial_\nu (A_\lambda \psi_\sigma)$. The charge can then be written as a surface integral at infinity:

$$S_\alpha = \lim_{r\to\infty} \int_S d\Omega \, r^2 \, n_\nu \, A^{ij}_\lambda \, (\gamma_5 \sigma_{ij} \psi_\sigma)_\alpha \, \varepsilon^{0\nu\lambda\sigma} \tag{22}$$

where $n_\nu = (0,\vec{n})$ is a unit vector normal to the sphere S. If one assumes that all the long range effects are already contained in the classical solution, the fields A'_μ and ψ'_μ will fall off at $r\to\infty$ faster than $1/r^2$ and will not contribute to the charge. We now assume that the field ψ_μ interacts with other fields which act as sources and for simplicity consider solutions of Einstein's equation outside the sources, which are static and asymptotically flat. Then the h^i_μ's must have asymptotically the Scharzschild form. Up to a gauge transformation we have for large r, $h^0_\mu \approx \delta^0_\mu (1-KM/r)$, $h^i_\mu \approx (\delta^i_\mu - KM\, x_\mu x^i/r^3)$ for $\mu \neq 0$ and $h^i_0 \approx 0$. Calculating a^{ij}_μ from eq. (16) and replacing the ansatz (14) in (22) one obtains:

$$S_\alpha = 8\pi \, KM \, \theta_\alpha \tag{23}$$

where K is the gravitational constant and M the mass of the background state. Thus, θ_α is related to the supersymmetry charge.

In order to determine the independent canonical variables in the theory we investigate small oscillations of the fields about a classical solution in a region of space-time far-removed from the sources. We then linearize the equations with respect to the primed variables and use the flat background metric $h^i_\mu = \delta^i_\mu$.

Since we have the local gauge invariance (9, 10) we can fix the gauge by $h_i^{\mu}\gamma^i\psi'_{\mu} = 0$. Then the equations of motion reduce to $\partial^{\mu}\psi'_{\mu} = 0$ and $\not{\partial}\psi'_{\mu} = 0$. The two conditions $\gamma\cdot\psi' = 0$, $\partial\cdot\psi' = 0$ remove the two spin-1/2 components of the spinor field, leaving a massless spin-3/2 Rarita-Schwinger field, the gravitino, as the only independent dynamical degree of freedom. Hence although classically we agree with ordinary gravity, there does not appear to be a graviton associated with a spin-2 elementary quantum oscillator in this theory. Gravitation arises as a classical solution corresponding to a collective effect of the fundamental spin-3/2 gravitinos.

The arguments presented here should become more precise in a fully quantized version of this theory, which must also include the other interactions and symmetries existing in Nature.

REFERENCES

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(3) The connection is a vertical form. The horizontal form A is actually a connection difference that is the difference between the connection and a trivial connection corresponding to pure gauge (zero curvature).

(4) The algebra corresponding to this transformation is based on OSp(1,2c) which is different from the conventional supersymmetry algebra. Its realization on asymptotic states may not be linear. This would be consistent with the absence of a spin-2 particle.