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DEPARTMENT OF PHYSICS   BERKELEY, CALIFORNIA 94720

March 11, 1981

Dear Sir,

Please find enclosed three copies of my entry for this year's Gravity Prize. It is entitled, "General Relativistic Chaos and Non-Linear Dynamics".

Yours sincerely,

John Barrow

John D. Barrow

JDB:cm

Biographical sketch:

John D. Barrow was born in London in 1952, read mathematics at the University of Durham (1971-4) before becoming a research student of D.W. Sciama at Oxford University. He received his D. Phil. in 1977, was Lindemann Travelling Fellow in 1977/8 and a Research Lecturer at Christ Church and the Department of Astrophysics, Oxford from 1977-80. He is currently a Miller Fellow at UC Berkeley. His interests include gravitation, non-linear dynamics, astrophysics, reading and track athletics.

General Relativistic Chaos

and

Non-Linear Dynamics

John D. Barrow
Department of Physics
University of California,
Berkeley, CA 94720, U.S.A.

Gravity Essay, 1981

Summary

We describe how new ideas in dynamical systems theory find application in the description of general relativistic systems. The concept of dynamical entropy is explained and the associated invariant evaluated for the Mixmaster cosmological model. The description of cosmological models as measure preserving dynamical systems leads to a number of exciting interconnections with new results in non-linear dynamics. This may provide a new avenue of approach to ascertaining the nature of the general solution to Einstein's equations.

Chaos is ubiquitous. During the last few years applied mathematicians and physicists from a variety of backgrounds have been intensively ivestigating the onset of chaotic behaviour in a wide spectrum of simple deterministic dynamical systems(1).

Until very recently if you had stopped a physicist in the street and asked him if deterministic systems could be chaotic he would have answered that 'random' behavior could only appear in the output of a dynamical system if its initial data were stochastic in nature or if some random forcing were coupled to it or a very large number of degrees of freedom excited. However, a series of studies have revealed that although any of these contingencies are sufficient pre-requisites for the appearance of random behaviour, none are in fact necessary. Very simple dynamical systems, notably iterated maps of the unit interval, with regular initial data, no stochastic forcing and very few degrees of freedom, exhibit behaviour which is for all practical purposes completely unpredictable(1,2).

Before describing the relevance of these ideas to general relativity we should explain the last statement; in particular, what is meant by 'chaotic' or 'random' behaviour in a deterministic system?

Consider a simple difference equation which rotates points around the circumference of a circle,(fig 1)

$$\theta_{n+1} = 2\theta_n \quad (\text{mod } 2\pi) \qquad (1)$$

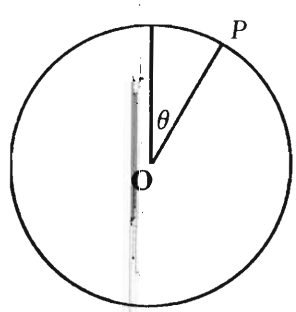

fig. 1

This discrete mapping is completely deterministic. If we know the initial position of P at θ_o precisely we will also know its subsequent position precisely after any specified number of iterations. However, suppose we approach (1) from a more realistic or "experimental" point of view. If (1) were a model of a real physical system our initial specification of θ_o will always be uncertain by some small amount $\delta\theta_o$. After the map has been iterated n times this small initial uncertainty will expand exponentially to fill a portion of the phase space (circumference) of angular extent

$$\delta\theta_n = 2^n \delta\theta_o \tag{2}$$

For sufficiently large n the finite initial uncertainty, however small, will eventually expand under the action of the mapping to fill the entire phase space. Although the mapping is completely deterministic any finite uncertainty in the initial data will render the output completely unpredictable after a sufficient number of iterations. In this essay we shall show that the Einstein equations are chaotic in this sense. Nearby trajectories in their solution space diverge from each other exponentially fast as they evolve with time and there is a breakdown of determinism quite distinct from that associated with either quantum effects or Cauchy horizons. To make this idea clearer we need to develop some further ideas:

Difference equations are easier to deal with than differential equations. How can we associate a difference equation with the Einstein equations?

If we take a homogeneous general relativistic cosmology[3] the Einstein equations will describe its evolution via a set of m ordinary differential equations

$$\dot{\underset{\sim}{x}} = F(\underset{\sim}{x}) \quad \underset{\sim}{x} = (x_1, \ldots x_m) \tag{3}$$

The solution of (3) corresponds to a trajectory (or flow) in some m dimensional phase space. We can place an (m-1) dimensional cross-section, Σ, through this phase space so that Σ is intersected infinitely often by the flow. The sequence of intersections that the flow makes with Σ gives a difference equation describing an induced mapping of Σ into itself[1]. This (m-1) dimensional difference equation is called the Poincare return map of the dynamical system (3). Clearly if (3) exhibits chaotic behaviour it will be mirrored in the behaviour of the associated Poincare map, whereas if its solution is periodic the flow in phase space is a closed orbit and the Poincare map would be simply a fixed point.

Our simple example, (1), possesses sensitive dependence on initial conditions as can be seen from its solution (2). To make this idea precise and more general we must also take into account the relative liklihood that a trajectory pass through one point of the phase space, or intersect one point of Σ, rather than another. To do this we require a smooth invariant measure for the flow or, in practice, for the return mapping. If the dynamical system given by the Einstein equations has two degrees of freedom, as is the case for the Mixmaster Universe we shall discuss below, then the Poincare map will be a one dimensional difference equation,

$$x_{n+1} = T(x_n) \tag{4}$$

If two solution trajectories are initially separated by $\Delta_o \equiv \delta x_o$ then their subsequent separation is represented by

$$|\Delta_t| = |\Delta_o| \int \frac{dx_t}{dx} \delta(x_o - x)\, dx \tag{5}$$

Suppose that the trajectory position at time t is given by x_t and is linked to the initial state by an operator $\mathscr{L}^t$ so

$$x_t = \mathcal{L}^t x \tag{6}$$

where $\mathcal{L}$ is defined by its action on any function $\psi(x)$

$$\mathcal{L}\psi(x) = \psi(T(x)) \tag{7}$$

We define the average rate of divergence of neighbouring trajectories as

$$h = \frac{\langle \log|\Delta_t|/|\Delta o|\rangle}{t} x_o \tag{8}$$

where <.....> is the expectation over the initial ensemble of possibilities (measure) for x_o; that is, "on the average" near-by trajectories in phase space diverge like e^{ht}.

If we set

$$K_t(x) \equiv t^{-1} \log \left|\frac{dx_t}{dx}\right| \tag{9}$$

then it is easy to show from (4)-(9) that

$$(t+1)K_{t+1}(x) = tK_t(x)+\log|T'(x)| \tag{10}$$

Now if we average this over the invariant measure, μ, for T then

$$h = \langle \log|T'(x)|\rangle \equiv \int \log|T'(x)|\mu(x)dx \tag{11}$$

The quantity $h(\mu,T)$ so constructed is the <u>metric (or Kolmogorov) entropy</u>[4] of the measure preserving map T. Formal chaos exists in a system when $h \neq 0$. Such systems, although deterministic, are not predictive because of their sensitivity to initial data. Information regarding the position of the solution trajectory is lost at each iterative step. If $\pi_n(y,x)$ is the joint probability that $T^n(x_o) = x$ and $T^{n-1}(x_o) = y$ then the information loss under T iteration is given by the Shannon formula[5] as

$$I(x) = -\sum_{y\varepsilon T^{-1}(x)} \pi(y,x) \log_2 \pi(y,x) \tag{12}$$

Taking the expectation of this information loss over the invariant measure $\mu(x)$ preserved by T we just obtain the metric entropy $h(\mu,T)$; that is

$$h(\mu,T) = \int I(x)\mu(x)dx \tag{13}$$

because

$$\pi(y,x) = |T'(x)|^{-1} \frac{\mu(y)}{\mu(x)} \tag{14}$$

Once it is established that a system possesses a non-zero entropy then many other powerful properties follow [6]; isomorphism can be established between systems of equal entropy and the relative degree of chaos in different systems compared[7].

The most dynamically general behaviour exhibited by gravitating systems is predicted to arise through intrinsically general relativistic aspects of space-time. Some of these are displayed by the spatially homogeneous Mixmaster Universe of Bianchi type IX. The essential field equations for this cosmological model are given by [8]

$$(\ln a^2)'' = (b^2 - c^2)^2 - a^4 \quad , \text{ et cycl.} \tag{15}$$

where $'' \equiv abc\partial_t$ and t is proper time. The expansion scale factors in orthogonal directions are a(t), b(t) and c(t).

It is possible to view the Mixmaster system (15) as a chaotic flow in a two dimensional phase space that possesses a one-dimensional Poincare return mapping. It is well-known that the Mixmaster evolution passes through an infinite sequence of states resembling the Kasner space-time (given by (15) with right hand side zero) as the singularity is approached for $t \to 0$. This recurrence enables the Mixmaster return mapping to be constructed on the unit interval and is given by the following mapping of [0,1] into itself,

$$x_{n+1} = T(x_n) = x_n^{-1} - [x_n^{-1}] \tag{16}$$

where [x] denotes the integer part of x. This mapping is everywhere unstable, $(|T'(x)|>1)$, possesses an infinite number of discontinuities and is displayed below in figure 2.

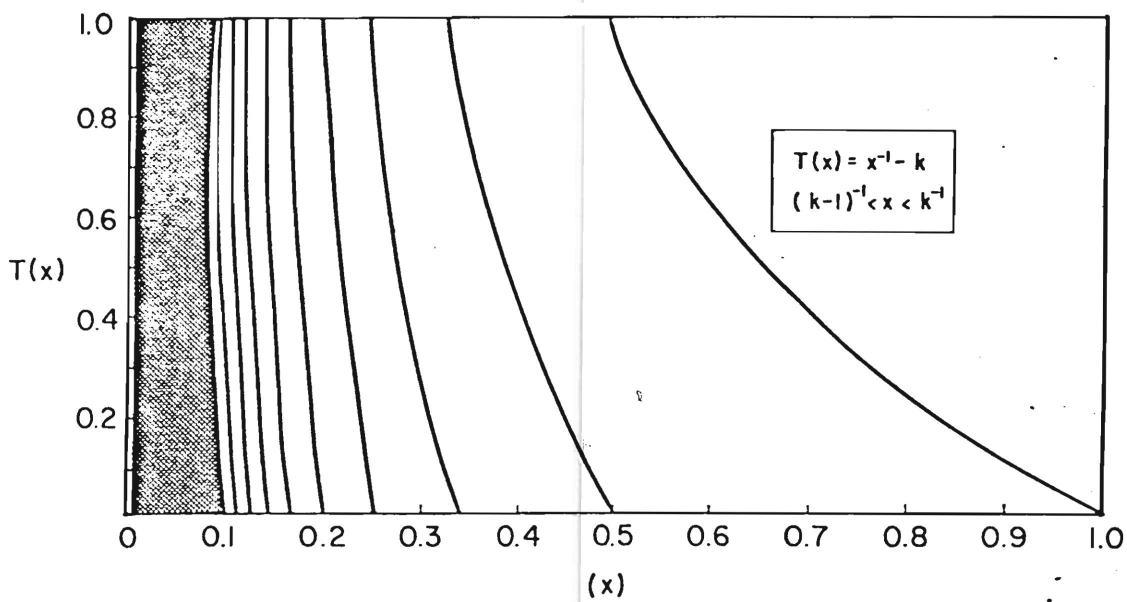

figure 2

Remarkably, T possesses an invariant measure that is smooth. That is, there exists a function μ_0 such that

$$\mu_0(\theta)\, d\theta = \sum_{k=1}^{\infty} \mu_0(x_k) dx_k \tag{17}$$

where

$$x_k = (\theta + k)^{-1} \tag{18}$$

The normalized measure is perfectly smooth and absolutely continuous with respect to Lebesgue measure (i.e. has the same sets of zero measure). It is just

$$\mu_o(\theta) = \frac{1}{(1+\theta)\ln 2} \tag{19}$$

Since $\inf |T'(x)| > 1$ and T is piecewise continuous T is also ergodic and it is straightforward to show it is isomorphic to a Bernoulli shift and therefore also strongly mixing[(11)].

The existence of μ_o allows the Mixmaster Universe to be viewed as a measureable dynamical system. Its metric entropy can be calculated from (11) and (16) as

$$h(\mu_o,T) = \int_0^1 \frac{\ln|T'(x)|}{(1+x)\ln 2}\,dx = \frac{\pi^2}{6(\ln 2)^2} = 3.4237... \tag{20}$$

On the average Mixmaster Universes evolving from neighbouring Cauchy data diverge ~ exp (3.4237t). A 'chaotic cosmology' can be rigorously defined as a solution to Einstein's equation whose dynamics possess a non-zero metric entropy.

Another way of viewing this dynamical invariant is as a measure of the rate of generation of distinct trajectories in a system over an infinite time with arbitrarily fine discrimination between trajectories. This is analogous to thermal entropy which is a measure of the number of configurations admitted by a system as the number of its degrees of freedom becomes infinite.

These results create an inter-face between general relativity and dynamical systems theory that appears to be potentially very fruitful. It is possible to classify homogeneous cosmological models according to the entropy of their return maps. What emerges is that, amongst vacuum models, only

Bianchi types VIII and IX have non-zero entropies and this is related to the presence of true gravitational degrees of freedom[9]. (As an aside we point out that types VIII and IX can be shown to be isomorphic to geodesic flows on negatively curved spaces that exhibit trajectory divergence and chaos[10]. Other homogeneous Universes, like the Kasner Universe, with purely kinematic degrees of freedom describe geodesic flows on flat spaces which do not exhibit trajectory divergence.)

There has been considerable speculation[11] regarding the possible existence of a 'gravitational entropy' associated with the gravitational field of an expanding Universe. It would monitor any deviation from exact isotropy and homogeneity and generalize the Bekenstein-Hawking[12] entropy to non-stationary space-times. We have shown that dynamical entropies can be calculated for cosmological space-times and they do indeed give a measure of the irregularity in the expansion dynamics in a straightforward and invariant manner. Penrose's[11] scenario might correspond to the evolution of a closed Universe from an almost Friedmannian initial state to a second singularity of Mixmaster character. Initially the Weyl curvature would be small and metric entropy zero; finally the Weyl curvature would become large and $h \neq 0$. Since the gravitational wave and curvature anisotropy create the chaotic Mixmaster dynamics they are also simply related to the Mixmaster metric entropy.

One last intriguing possibility is the hint of a new approach to analysing the general solution of the Einstein equations near a space-time singularity. A large class of non-linear difference equations have been shown to possess remarkable universal properties[1,13], independent of their precise form. If it could be established that general solutions to the Einstein equations possess Poincare maps with some universal characteristics, then important deductions could be made about the nature of the general solution without knowing it in closed form. In effect one is viewing the general solution to the Einstein equations as a description of gravitational turbulence and anticipating that this turbulence develops, like hydrodynamic turbulence, by a process of period doubling(13) (note this is not the case for (16)). Two simple examples should give the idea: suppose one examined an inhomogeneous space-time close (in some sense) to the Mixmaster model. One imagines that the return mapping for this dynamics might also be "close" to that for the Mixmaster universe. Suppose for illustrative purposes, it differed by some small constant ε so

$$x_{n+1} = T(x_n) = x_n^{-1} - [x_n^{-1} + \varepsilon] \quad , \; \varepsilon > 0 \tag{21}$$

Then a smooth measure μ_ε still exists for T and the metric entropy is calculated to be

$$h(\mu_\varepsilon, T) = \frac{\pi^2}{6(\ln 2)^2} \cdot \left(\frac{\ln 2}{\ln(2-\varepsilon)} \right) \quad , \; 0 < \varepsilon < \frac{3-\sqrt{5}}{2} \tag{22}$$

As ε increases the entropy does also, although in this example only because it becomes more probable that trajectories get close to highly unstable points near $x \approx 0$. This example is not intended to be very realistic although it is possible to relate ε to the 'error' introduced by viewing cycles of Mixmaster evolution as undergoing periods of small oscillations from one exact Kasner model to another(7).

Since the non-zero metric entropy of Mixmaster arises entirely from the cycle to cycle evolution (and not the small oscillations within a cycle where $T' = 1$) the most interesting perturbations, or generalizations of the Mixmaster Universe, would alter the functional form of the Poincare map, say to

$$T(x) = F(x) - [F(x)] \tag{23}$$

or some higher dimensional analogue, where F is a C^r function ($r > 2$).

It is known[14] that a C^{r-1} invariant measure always exists for[23] and so the metric entropy can always be evaluated. It remains to be seen whether these exciting possibilities can be realized.

We have shown how new ideas in non-linear dynamics provide a natural description of the most general behaviour so far detected in the Einstein equations. This enables new invariants to be calculated for complicated space-times and allows the concepts of 'chaotic cosmology' and 'gravitational entropy' to be made rigorous. Finally, we should add, it creates a new field of enquiry for dynamicists. A field that should prove exceptionally fertile, for the unique self-interacting non-linearity of general relativistic systems hints at the presence of chaotic behaviour of unsuspected subtlety.

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REFERENCES

1. D. Ruelle, F. Takens, Comm. Math. Phys. 20, 167 (1971) P. Collet, J-P. Eckmann, 'Iterated Maps on the Interval as Dynamical Systems', (Birkhauser, Boston 1940)

2. D. Ruelle, Lecture Notes in Physics 80, (Springer-Verlag, Berlin, 1978).

3. M.A.H. MacCallum, in 'General Relativity: An Einstein Centenary' ed S.W. Hawking, W. Israel (CUP, 1979).

4. R. Bowen, Am. Math. Soc. CBMS No. 35 (1977)
   J.B. Pesin, Russian Math. Surveys 32, 55 (1977)

5. C. Shannon, W. Weaver, 'The Mathematical Theory of Communication' (Univ. of Illinois, 1962); R. Shaw 'Strange Attractors, Chaotic Behaviour and Information Flow', UC Santa Cruz (1980).

6. Y. Oono, M. Osikawa, Prog. Theo. Phys. 64, 54 (1980).

7. D. Ornstein 'Ergodic Theory, Randomness and Dynamical Systems', Yale Math. monographs No. 5.

7. V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, Adv. in Physics 19, 525 (1970)
   C.W. Misner, Phys. Rev. Lett. 22, 1071 (1969)
   A.D. Doroshkevich, E.D. Novikov, Sov. Astron. 14, 763 (1971)

8. See Bowen (ref 4).

9. R. Jantzen, Ann. Inst. Poincare, 23, 121 (1980)

10. D.V. Anosov, J.G. Sinai, Usp. Mat. Nank. 22, 107 (1967)

11. R. Penrose in 'General Relativity: An Einstein Centenary' ed S.W. Hawking, W. Israel (CUP 1979)

12. S.W. Hawking, Comm. Math. Phys. 43, 189 (1975)

13. M. Feigenbaum, J. Stat. Phys. 19, 25 (1978); 21, 669 (1979); Comm. Math Phys. 77, 65 (1980)

14. A. Renyi, Acta. Math. Acad. Sci. Hung. 8, 477 (1957)