

# Gravity's universality: The physics underlying Tolman temperature gradients

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ABSTRACT:

We provide a simple and clear verification of the physical need for temperature gradients in equilibrium states when gravitational fields are present. Our argument will be built in a completely *kinematic* manner, in terms of the *gravitational red-shift/blue-shift* of light, together with a relativistic extension of Maxwell's two column argument. We conclude by showing that it is the *universality* of the gravitational interaction (the uniqueness of free-fall) that ultimately permits Tolman's equilibrium temperature gradients without any violation of the laws of thermodynamics.

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DATE: 23 March 2018, LaTeX-ed March 23, 2018.

PACS: 04.20.-q; 04.40.-b; 05.20.-y; 05.70.-a

*Essay written for the Gravity Research Foundation 2018 Awards for Essays on Gravitation.*

Gravity is universal. That, for sure, is something that you will come across when learning general relativity. Einstein’s *gedankenexperiment*, with the observer inside an elevator in a sufficiently long free-fall, having time to experience her own limited spacetime, just like Alice falling down the rabbit hole and wondering about the universe, is a really inspiring and surprisingly useful way to do physics. We will apply some of that way of thinking in this essay.

The question we will like to ask and answer in here is whether all the consequences of this universal character of gravity have already been fully explored. We believe that they have not. A major goal of this essay will be to simply and clarify the ideas behind already existent and well known phenomena. Maybe to show a new interpretation and way to see the world, more precisely, thermodynamics. We aim to arrive at the end, to the conclusion that general relativity has a lot to add to classical thermodynamics, in the same way that thermodynamics can change our way to see some gravitational effects. Neither black holes nor horizons nor fire-walls will be present. The systems we will be analyzing are simply boxes full of gases in a pre-defined version of thermal equilibrium.

We can start by defining *thermal equilibrium*. Textbooks on non-relativistic thermodynamics normally present something along these lines: If a system is in thermal equilibrium, its temperature field  $T(t, x_i)$  will be a constant,  $t$  being the proper time of the observers at rest with respect to the fluid and  $x_i$  the spatial components in the chosen coordinate system. On the other hand, relativistic thermodynamics already understands that this definition is somewhat misleading when gravity is present.

Relativistic thermal equilibrium in a gravitational field has the interesting feature that the locally measured temperature  $T(x_i)$ , that is, what a physical thermometer would measure, has a small non-zero spatial gradient [1, 2]<sup>1</sup>. Indeed

$$\frac{\nabla T(z)}{T(z)} = -\frac{g}{c^2} \tag{1}$$

in the flat-earth approximation, with  $g$  being the local gravity acceleration and  $c$  the speed of light. Near the surface of the Earth this gradient is approximately  $10^{-16}\text{m}^{-1}$ ; certainly negligible in most experimental settings. However, the fact that this is non-zero implies that there must be subtle revisions in the non-relativistic versions of the zeroth law, as well as the Clausius version of the second law, and the non-relativistic Fick’s law [3].

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<sup>1</sup>Somewhat confusingly, in Tolman’s original articles, he uses  $T_0$  to denote the locally measured position-dependent temperature — which we denote  $T(x_i)$ . In contrast, we reserve  $T_0$  for the redshifted temperature, which is spatially constant in thermal equilibrium.

The Tolman result was first established in general relativity [1, 2], where in any *static* gravitational field, that is, a spacetime metric which can be put in the block-diagonal form<sup>2</sup>:

$$ds^2 = -N(x_i)^2 dt^2 + g_{ab} dx^a dx^b, \quad (2)$$

one can show

$$T(x_i) = \frac{T_0}{N(x_i)}; \quad T_0 = T(x_i) N(x_i). \quad (3)$$

In this way, we see that we have two distinct temperature functions. One is the locally measured, position-dependent  $T(x_i)$  Tolman temperature. The second,  $T_0$ , represents the red-shifted temperature.<sup>3</sup> It is this  $T_0$  temperature, which is spatially constant in thermal equilibrium, that drives the direction of heat flow in the Clausius version of the second law, and it is gradients in this temperature that drive the modified version of Fick's law:

$$(\text{heat flux}) \propto \nabla T_0 = \nabla (T(x_i)N(x_i)). \quad (4)$$

What we want to argue now is that, although extremely well written and didactic, Tolman's work [1] creates the impression that the necessary inputs to obtain the temperature gradient are considerably more complex and demanding than they actually are. Stripped to its essence, Tolman's argument can be rephrased in a completely *kinematic* manner in terms of the *gravitational red-shift/blue-shift* of light.

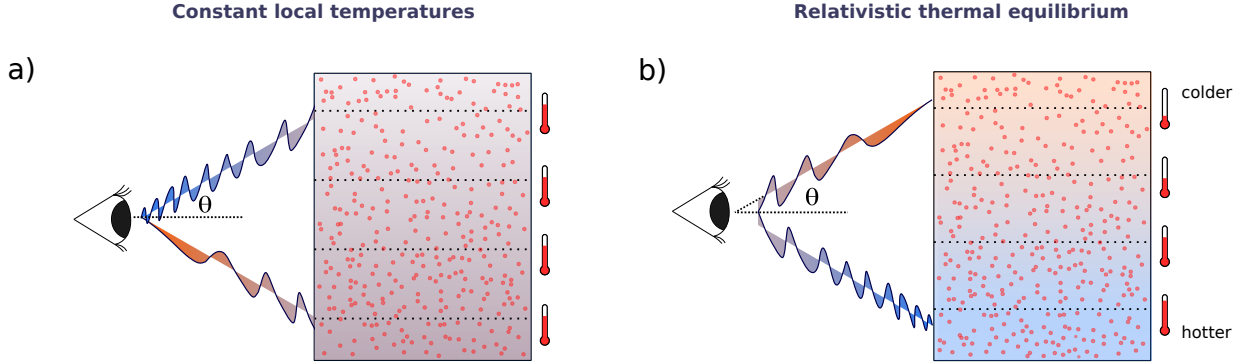
Let us further explore the meaning of these words. Consider a static observer looking at an angle  $\theta$  (with the horizontal) to a photon gas in which all local thermometers measure the same local temperature  $T(x_i)$  (see figure 1-a). As it is well known, if a photon is travelling along the  $\theta$  direction from some distance  $r$ , it would be coming from a height  $z = r \sin \theta$  and would suffer a red-shift/blue-shift of a factor  $(1 + gz/c^2)$  by the time it finally arrives at the observer.

On the other hand, photons coming from different gravitational equipotential slices suffer distinct gravitational blue-shifts/red-shifts. In this way, if the locally measured temperatures  $T(x_i)$  are spatially constant, the observer will see not a simple Planck spectrum, but rather a *superposition* of Planck spectra of different temperatures.

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<sup>2</sup>In terms of the gravitational potential  $N(x) \approx 1 + \Phi(x) \approx 1 + gz/c^2$ .

<sup>3</sup> $T_0$  is the physical temperature at the place where the gravitational potential  $\Phi$  is chosen to be zero. For instance, if we normalize so that  $\Phi = 0$  at  $z = 0$ , then  $T_0$  is just the physical temperature at  $z = 0$ . On the other hand, if we normalize so that  $N \rightarrow 1$  at spatial infinity, then  $T_0$  is the physical temperature at spatial infinity.



**Figure 1.** External observer looking at photons leaking from: a) the box containing the photon gas with constant temperature; and b) the box with the photon gas in relativistic thermal equilibrium.

But in this case, the system is not at thermal equilibrium from the point of view of the observer; we have a *reductio ad absurdum*. The only way to avoid inconsistency is taking into account Tolman’s temperature gradients, i.e. to have a position-dependent temperature  $T(x_i) = T_0/(1 + gz/c^2) \approx T_0 (1 - gz/c^2)$ , since then the gravitational red-shift/blue-shift will *exactly* cancel the gradient factor from the temperature, guaranteeing that all the Planck spectra photons, when seen by the static observer, will be perceived to have the *same* temperature  $T_0$  (see figure 1-b).

A crucial point in here is that, although it is important that an external observer will perceive a constant temperature when looking at the box, the fact the photons *inside* the box will inevitably suffer blue-shifts/red-shifts when moving around is what actually proves the necessity for temperature gradients in equilibrium states in the presence of gravitational fields.

Now, coming back to gravity’s universality, how are these ideas related? The fact is that, without gravity’s universal character, Tolman’s temperature gradient would indeed violate the second law of thermodynamics. The first to notice this was James Clerk Maxwell [4, 5], as can be seen in the following argument regarding the equilibrium temperature of a vertical column of gas:

“[...] if two vertical columns of different substances stand on the same perfectly conducting horizontal plate, the temperature of the bottom of each column will be the same ; and if each column is in thermal equilibrium of itself, the temperatures at all equal heights must be the same. In fact, if the

*temperatures of the tops of the two columns were different, we might drive an engine with this difference of temperature, and the refuse heat would pass down the colder column, through the conducting plate, and up the warmer column; and this would go on till all the heat was converted into work, contrary to the second law of thermodynamics.”*

Its clear from this argument that horizontal (equilibrium) temperature differences cannot exist at equal heights. They will necessarily drive heat fluxes and would allow the construction of a *perpetuum mobile*. In that manner, the vertical temperature gradients, if present at all, must be the same in both columns, regardless the material they are made of. In other words, they must be *universal*. This is a perfectly valid generally correct statement. However, Maxwell’s original argument actually has a continuation:

*“But we know that if one of the columns is gaseous, its temperature is uniform [from the non-relativistic kinetic theory of gases]. Hence that of the other must be uniform, whatever its material.”*

This final conclusion, as we now know, ends up not being valid relativistically<sup>4</sup>. Given the radiation gas discussion above, we can even rewrite a relativistic version of this final conclusion as:

*But we know that if one of the columns is a photon gas, its temperature must be position dependent, as given by Tolman’s relation. Hence that of the other must be position dependent as well, whatever its material.*

In this way, Tolman’s temperature gradient can be proved with simple arguments for any kind of material or physical state of matter.

But, what about the other forces, *e.g.* electromagnetism? Are equilibrium temperature gradients in an electron gas, for example, completely ruled out? Well, yes [3]. To prove this simply consider an electron gas, inside a box, submitted to an external electric field. Admit thermal equilibrium. Assume also a very low gas density, so that the force exerted by the external field is much stronger than the interactions between electrons. If a temperature gradient occurs, it will be aligned with the electric field direction. No gravitational field is present.

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<sup>4</sup>It is important to point out that, in the non-relativistic limit, Maxwell is actually correct, given that  $\nabla T(z) \rightarrow 0$  for  $c \rightarrow \infty$ , as can be seen from equation (1).

In case a temperature gradient exists after equilibrium is reached, we can, for example, place next to the just mentioned electron gas a box filled with electrically neutral particles, *i.e.* photons, neutrons, *etc.* Due to its neutrality, it will not interact with the electric field, thus having no reason at all to present a temperature gradient. Continuing the argument on the same lines as Maxwell did, we can see that this situation would indeed generate heat flows, enabling a perpetual motion machine of the second kind.

The validity of this argument comes from the non-universal character of electric forces. All we needed being two different compositions for the columns, one which reacts to the electric field (charged particles) and one that doesn't. But that machinery can easily be extended to any force which is not universal, allowing us to say:

*Given that temperature gradients created by any force that is not universal (e.g. dependent on charge, mass, spin,...) allow the creation of heat machines that violate the second law, these temperature gradients must not exist.*

Going even further, up to date no force other than gravity seems to act on all sources of mass or energy in the same way regardless, so we can even state this as:

*Gravity is the only force capable of creating temperature gradients in equilibrium states without violating any law of thermodynamics.*

## Acknowledgements

JS was supported by a Victoria University of Wellington PhD Scholarship. JS wishes to thank Cesar Uliana-Lima and Uli Zuelicke for helpful discussions. MV was supported by the Marsden Fund, administered by the Royal Society of New Zealand.

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