Einstein-Hilbert action, with quantum corrections, from the Planck scale coarse-graining of the spacetime microstructure*

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Abstract

The gravitational dynamics of the coarse-grained spacetime geometry should emerge from extremizing the number of microscopic configurations, Ω, of the pregeometric variables corresponding to a given geometry. This Ω will be the product over all events $\mathcal{P}$ of the density, $\rho(\mathcal{P})$, of microscopic configurations associated with each event $\mathcal{P}$. I show how ρ can be computed, in terms of the Van-Vleck determinant, and thus obtain directly the gravitational effective action $\mathcal{L}_E$ at mesoscopic scales. The leading term of this, non-perturbavive, effective action gives the Einstein-Hilbert action, thereby providing its microscopic derivation. The higher order corrections are finite without any need for regularisation and I demonstrate how they can be computed in a systematic manner.

The nature of spacetime dynamics allows us to define three characteristic regimes: macroscopic, mesoscopic and microscopic. A complete model for the quantum spacetime is needed to probe *microscopic* length scales $\lambda \lesssim L_P$ (where $L_P^2 = G\hbar/c^3$ is the Planck length) while the classical Einstein equation is adequate for probing *macroscopic* length scales $\lambda \gg L_P$. At the intermediate *mesoscopic* scales which are large — but not significantly large — compared to L_P, one would expect the spacetime to be still described in terms of an *effective* geometry, determined by the dynamical equations which incorporate the quantum gravity (QG) corrections. I will describe how we can study this regime in terms of a suitable extremum principle which, at the leading order, will reduce to that based on the Hilbert action. This can be done by combining classical geometric considerations with coarse-graining over sub-Planckian degrees of freedom.

An analogy between spacetime dynamics and fluid dynamics will be helpful to set the stage. In the study of a fluid made of discrete atoms ("microscopic degrees of freedom") one can again delineate three levels of description. The first one is *microscopic* in which one would describe the system in a completely quantum mechanical language by, say, writing the Schrodinger equation for all the atoms of the fluid. At the other extreme, we have a *macroscopic* description, in terms of continuity and Navier-Stokes equations,

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which *totally ignores* the granularity of the fluid and treats it as a continuum. In this regime we would, for example, describe the velocity of the fluid at a given event x^i by a *single-valued* vector function $\boldsymbol{v}(t,\boldsymbol{x})$. These two descriptions are analogous to the microscopic and macroscopic descriptions of the spacetime I mentioned earlier. What we are looking for is a mesoscopic scale description interpolating between these two. The kinetic theory of fluids, in terms of a distribution function $f(x^i,p_i)$, provides such an interpolation between these two extremes. The distribution function recognizes the discreteness of the fluids — with $dN=f(x^i,p_i)\,d\Gamma$ counting the *number* of atoms in a phase volume $d\Gamma\equiv d^3\boldsymbol{x}\,d^3\boldsymbol{p}$ — and, at the same time, allows a continuum description at scales sufficiently larger than the mean free path. In this mesoscopic scale description we recognize the existence of velocity dispersion at any given event x^i due to discrete atoms with different momenta p_i co-existing at a given event. What we are interested in is a similar description, at mesoscopic scales, for the spacetime fluid with Planck length playing the role of the mean free path.

To obtain such a description, we can proceed as follows: Consider a spacetime geometry $\mathcal{G}$, described a metric tensor $\bar{g}_{ab}(\bar{x})$ in an arbitrary coordinate chart $\bar{x}^i$. There will a large number, Ω, of microscopic configurations of (as yet unknown) pre-geometric variables which will be consistent with a given emergent geometry $\mathcal{G}$. (This is similar to the existence of several microscopic configurations of the atoms of a fluid, consistent with some macroscopic parameters like pressure, density etc). Let $f(\mathcal{P},\xi_A)$ be the number of possible configurations of the microscopic degrees of freedom of spacetime, associated with an event $\mathcal{P}$, in a given geometry $\mathcal{G}$. As indicated, this function $f(\mathcal{P},\xi_A)$ could also depend on the relics of some sub-Planckian degrees of freedom, symbolically denoted by ξ_A. (These are analogous to the momenta of individual atoms in kinetic theory of a fluid.) The total number of microscopic configurations in a given region of spacetime is then given by

$$\Omega\equiv\prod_{\mathcal{P},\xi_A}f(\mathcal{P},\xi_A)=\exp[\sum_{\mathcal{P},\xi_A}\ln f(\mathcal{P},\xi_A)]\Rightarrow\exp\left[-\int d^4\bar{x}\,\sqrt{\bar{g}}\,\mathcal{L}_E\right]\equiv\exp -A_E \tag{1}$$

where

$$\mathcal{L}_E=-L^{-4}\sum_{\xi_A}\ln f(\mathcal{P},\xi_A)\equiv -L^{-4}\ln\rho(\mathcal{P});\qquad \rho(\mathcal{P})\equiv\prod_{\xi_A}f(\mathcal{P},\xi_A) \tag{2}$$

Here $\rho(\mathcal{P})$ denotes the number of microscopic configurations associated with a given event $\mathcal{P}$ once the product over internal variables ξ_A is taken. In the third step in Eq. (1), we have taken the continuum limit and introduced the length-scale $L=\mathcal{O}(1)L_P\equiv\mu L_P$, of the order of Planck length, to ensure proper dimensions.[1] The A_E can be thought of as an effective (Euclidean) action. The QG corrected field equations at mesoscopic scales can then be obtained by extremising the microscopic configurations $\Omega=e^{-A_E}$ with $\mathcal{L}_E$ playing the role of a gravitational effective Lagrangian.

Our next task is to determine $f(\mathcal{P},\xi_A)$ and ρ. As we will soon see, the variables ξ_A arise very naturally — and automatically — in this approach. So, let me ignore

[1] So, the discrete sum over events $\mathcal{P}$ goes over to integration with dimensionless proper volume measure $\sqrt{\bar{g}}d^4x/L^4$ in the continuum limit.

these variables ξ_A just for the moment so that f is the same as ρ. Then I would expect $f = \rho \propto \sqrt{g}$ so that $\rho(\bar{x})d^4\bar{x} \propto \sqrt{g}(\bar{x})d^4\bar{x}$ will scale as the proper volume of a region. The QG corrected field equations of the gravitational sector has to then come from the effective Lagrangian $\mathcal{L}_E = -L^{-4}\ln\sqrt{g}$ when we vary the metric. At first sight, this seems impossible because of three issues: (i) We would expect the extremum principle to be based on the Ricci scalar R at the appropriate limit rather than on $\ln\sqrt{g}$. (ii) One would like to interpret $\sqrt{g}$ in terms of a suitable scalar quantity, after eliminating the gauge freedom in the metric tensor at any event $\mathcal{P}$. (iii) One cannot expect to get a finite number of microscopic degrees of freedom purely classically, without invoking some QG considerations. I will now show how all these issues can be tackled with the proper interpretation of the $\sqrt{g}$ factor.

The metric tensor $\bar{g}_{ab}(\bar{x})$ (and its determinant), at any given event $\mathcal{P}'$, contain come amount of gauge redundancy in any generic coordinate system. So our first task is to eliminate this freedom at an event by choosing an appropriate coordinate system. Such a coordinate system, usually called Riemann normal coordinates, is well known in literature (see e.g., [1,2]). If one introduce the Riemann normal coordinates (RNC), centered around an event $\mathcal{P}'$ in the spacetime, the new metric $g_{ik}(\mathcal{P},\mathcal{P}')$ at any another event $\mathcal{P}$ will depend on *both* $\mathcal{P}$ and $\mathcal{P}'$. That is, in the RN coordinates, we get a *family* of metrics $g_{ik}(x,x')$ at x, parameterized by the coordinates of another event x'. The construction of RNC ensures that metric reduces to the Cartesian form, and the Christoffel symbols vanish, at x'.

This choice also allows a co-ordinate invariant description. It is well-known that [3], the $\sqrt{g}(x,x')$ in RNC is precisely equal to the reciprocal of the Van-Vleck[2] determinant, $\Delta^{-1}(x,x')$, obtained from the second derivatives of the geodesic interval, $\sigma^2(x,x')$. That is, $\Delta^{-1}(x,x')$ is a (bi)scalar which reduces to $\sqrt{g}$ in RNC; so by using $\Delta^{-1}(x,x')$ in arbitrary coordinates, we also obtain a covariant description.[3] The metric determinant $\sqrt{g} = \Delta^{-1}$ in RNC can now be expressed in terms of the geometrical variables built from the curvature tensor R_{ijkl} and its derivatives. It will also depend on the geodesic distance $\sigma^2(x,x')$ and the unit-norm vectors $n_i(x,x') \equiv \partial_i\sigma$ with the latter two appearing (only) through the combination $q^i(x,x') \equiv \sigma n^i$. We can, therefore, write down the functional

$$\Delta^{-1}(x,x') = (\sqrt{g})_{RNC} = f[R_{ijkl}(x);\sigma n^i] = f[x;q^i(x,x')] \tag{3}$$

where it is understood that the functional dependence on the curvature also includes dependence on its derivatives.[4]

This expression is non-local (i.e., it depends on two events x and x') while we are looking for a local expression for the density of microscopic degrees of freedom associated with an event. The simplest procedure to get a local expression will be to take the limit $x \to x'$; this, however, does not work because — in this limit — $\sigma(x,x)$ vanishes and all the dependence on the curvature disappears. This, of course, is to be expected.

[2]The Van-Vleck determinant is defined, in *arbitrary* coordinates, by: $\Delta(x,x') = D(x,x')/(\sqrt{g(x)}\sqrt{g(x')})$ where $D = (1/2)\mathrm{Det}[\partial_a\partial'_b\sigma^2(x,x')]$

[3]This trick of using Δ^{-1} to replace $\sqrt{g}$ in RNC in order to obtain a covariant description is well known in literature. See, e.g., [4,5].

[4]One could have introduced a proportionality constant λ between $(\sqrt{g})_{RNC}$ and f and written $f = \lambda\sqrt{g}$. *Observations* show that $\lambda = 1$ to a very high order of accuracy! I will discuss this later. The choice $\lambda = 1$ is equivalent to the normalization $\rho = 1$ in flat spacetime.

It will be impossible to obtain a non-trivial, local, measure of microscopic degrees of freedom without introducing some kind of 'discreteness' in the spacetime at Planck scales. A point of strictly zero size cannot host degrees of freedom in spacetime, just as it cannot host a finite number of atoms in a fluid. In a fluid we need to average over a volume which is large compared to the mean free path; similarly, in the spacetime we need to coarse-grain over Planck-size regions. This is most easily done by averaging (coarse-graining) $\ln\Delta(x^i,\sigma n^i)$ over the sub-Planckian 4-sphere S^4 by integrating the displacement $q^i\equiv\sigma n^i$ over a *4-sphere* of Planck scale radius L_0, say. This procedure is conceptually equivalent to replacing σ by L (which is related to L_0 by an unimportant numerical factor) and averaging over all n^i over a 3-sphere in the expression $\ln\Delta(x^i,\sigma n^i)$. We can now identify the density of microscopic degrees of freedom as $f[R_{ijkl}(x);\sigma n^i]\equiv\Delta^{-1}[R_{ijkl}(x);q^i]$, with q^i playing the role of internal variables ξ_A in Eq. (1). That is, a natural set of internal degrees of freedom ξ_A arises in the form of Planck scale displacement vector field q^i which are summed over. We thus get the final expression for the Euclidean effective Lagrangian:

$$\mathcal{L}_E=-L^{-4}\ln\rho[R_{ijkl}(x)]=L^{-4}\left[\int_{S^4}d^4q\ln\Delta[R_{ijkl}(x);q^i]\right] \tag{4}$$

This is a well-defined, non-perturbative, expression for the gravitational effective action for a coarse-grained spacetime.

The expression for effective Lagrangian Eq. (4) can be evaluated non-pertubatively if we know the exact dependence of $\ln\Delta$ on σ^2. Since we do not know this, the effective Lagrangian has to be computed as a series in L_P^2. This is easily done using the known [6,7] series expansions, for, say, g in RNC (with $q^i\equiv\sigma n^i$)

$$g=1-A_{ij}(q^iq^j)-A_{ijk}(q^iq^jq^k)-A_{ijkl}(q^iq^jq^kq^l)+\mathcal{O}(\sigma^5) \tag{5}$$

where $A_{ij}=(1/3)R_{ij}, A_{ijk}=(1/6)\nabla_kR_{ij}$ and $A_{ijkl}=(1/180)[9\nabla_l\nabla_kR_{ij}+2R^p{}_{ijq}R_{pkl}{}^q-10R_{ij}R_{kl}]$. Computing $(-\ln\sqrt{g})$, correct to $\mathcal{O}(\sigma^6)$ and integrating over all q^i inside a Planck scale 4-sphere, we get the effective Lagrangian in Eq. (4) to be:

$$\mathcal{L}_E=L^{-4}\left[\int_{S^4}d^4q\ln\Delta[R_{ijkl}(x);q^i]\right]=\frac{\pi^2}{12L^2}\left[R+L^2Q+\mathcal{O}(L^4)\right] \tag{6}$$

where $L=\mu L_P$ and

$$Q\equiv\frac{1}{2}\left[\frac{1}{20}\Box R+\frac{1}{90}R_{ab}R^{ab}+\frac{1}{10}\nabla_a\nabla_bR^{ab}+\frac{1}{60}R_{abcd}R^{abcd}\right] \tag{7}$$

The coefficient of R has to be fixed by comparing with the Newtonian limit, in terms of G_N by setting the coefficient to $(16\pi L_P^2)^{-1}$; this leads to $\mu^2=4\pi^3/3$. So our final result for the effective Lagrangian, correct to $\mathcal{O}(L_P^4)$, is:

$$\mathcal{L}_E=\frac{1}{16\pi L_P^2}\left(R+\frac{4\pi^3}{3}L_P^2Q+\mathcal{O}(L_P^4)\right) \tag{8}$$

These expressions allow us to compute quantum corrections to Einstein's equations in a systematic manner. Note that the corrections are finite and requires no additional regularization. *The conceptual simplicity of the approach which has led to such a tangible result is note worthy.*

Let me now comment on several aspects of this result.

The effective Lagrangian in Eq. (8) has no cosmological constant term because we took $f = \sqrt{g}$. If we had taken, instead, $f = \lambda\sqrt{g}$ with a proportionality constant λ, then the $\ln f$ in Eq. (4) would have contributed an extra $\ln\lambda$ terms appearing as a dimensionless cosmological constant $\Lambda L^2 \simeq \ln\lambda$. This tells you that $\lambda = 1 + \mathcal{O}(10^{-123})$. *That is, the natural choice* $\lambda = 1$ *indeed predicts zero cosmological constant.* It also suggests that the observed cosmological constant in the universe is a non-perturbative relic [8] from QG which changes λ by a tiny amount from unity.

The RNC is constructed so that it is locally inertial at a given event eliminating gauge degrees of the metric *at that event.* A synchronous reference frame with line element $ds^2 = d\sigma^2 + \sigma^2\gamma_{\alpha\beta}(\sigma, x^\alpha)dx^\alpha dx^\beta$, on the other hand, imposes the gauge conditions $g_{00} = 1, g_{0\alpha} = 0$ in a local *region* and serves the same purpose. It uses the geodesic distance σ as a "radial" coordinate. In this frame $\Delta^{-1} = \rho = (\gamma/\gamma_{flat})^{1/2}$, where $\sqrt{\gamma}$ is the unit area element of $\sigma =$ constant surfaces.[5] The density of microscopic degrees of freedom can now be thought of as being proportional to the dimensionless *area measure* $\sqrt{\gamma}$. The spatial surfaces $\sigma=$ constant maps to equi-geodesic surfaces $\sigma(x, x') =$ constant in an *arbitrary* frame. It is also well known [3] that the density of geodesics is given by the reciprocal of the Van-Vleck determinant $\Delta^{-1}(x, x')$. Our results suggest identifying the microscopic degrees of freedom of spacetime at the event $\mathcal{P}$ as the the density of geodesics, with suitable coarse-graining as we approach the Planck scales.

Our approach has opened up several further avenues of exploration. We now have an explicit expression for the density of microscopic degrees of freedom:

$$f[x^i, p_i] = \Delta^{-1}[R_{ijkl}(x), Ln^i]\,; \qquad n^i n_i = 1 \tag{9}$$

where $q^i = \Delta x^i \equiv Ln^i$ are Planck scale displacements in random directions. (The relation $f = \Delta^{-1}$ was first introduced in [9] and was explored further in the context of area of equi-geodesic surfaces in several previous works.) Here I have ignored all the dynamics contained in n^i and have merely summed over it. The next step would be to investigate whether one can obtain an evolution equation for $f(x^i, n^i)$ as in the case of fluid kinetics. The occurrence of the factor $\ln\Delta = \ln\mathrm{Det}\,\Delta_{ab'}$, where $\Delta_{ab'}$ is the Van Vleck matrix, suggests some possibilities. This factor will arise, for example, in a path integral over a vector field v^a of the amplitude $\exp[-v^a\Delta_{ab'}v^{b'}]$. All these signals the deep role played by the geodesic distance $\sigma^2(x, x')$, Van Vleck matrix $\Delta_{ab'}$ and its determinant Δ in the *microscopic* structure of spacetime.

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[5]The Δ satisfies the equation $\partial_\sigma(\ln\Delta) = 3\sigma^{-1} - K$ in this frame, where $K = \partial_\sigma \ln\sqrt{h} = \partial_\sigma \ln\sqrt{\sigma^3\gamma}$ is the extrinsic curvature of the equi-geodesic surfaces. This integrates to give $\rho = \Delta^{-1} \propto \sqrt{\gamma} = \sqrt{\gamma/\gamma_{flat}}$ with proper normalization.

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