Principle of Equivalence at Planck scales and the zero-point-length of spacetime *

— A synergistic description of quantum matter and geometry —

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Abstract

At mesoscopic scales close to, but somewhat larger than, Planck length one could describe quantum spacetime and matter in terms of a quantum-corrected geometry. The key feature of such a description is the introduction of a zero-point-length into the spacetime. When we proceed from quantum geometry to quantum matter, the zero-point-length will introduce corrections in the propagator of matter field in a specific manner. On the other hand, one cannot ignore the self gravity of matter fields at the mesoscopic scales and this will also modify the form of the propagator. Consistency demands that, these two modifications coming from two different directions, are the same. I show that this non-trivial demand is actually satisfied. Surprisingly, the principle of equivalence, operating at Planck scales, ensures this consistency in a subtle manner.

The mesoscopic regime of the spacetime interpolates between the microscopic regime, very close to Planck scale (which requires a full quantum gravitational description) and macroscopic regime, far away from the Planck scale (at which one can use the formalism of quantum field theory in *fixed* curved spacetime). This regime is close, but not too close, to the Planck scale so that we can still introduce some kind of effective geometric description incorporating dominant quantum gravity effects.

There are *two* distinct features which come into play in the mesoscopic regime, as we approach the Planck scale. The first, which is well-recognized, is the fact that spacetime close to Planck scales needs to be described very differently from spacetime at macroscopic scales. Much of the work in the area of quantum gravity, indeed, has something to say about this issue. The second feature — which has not been equally emphasized — concerns the matter sector: How do you describe matter — say, an electron — close to Planck scales? This question is non-trivial because no field — even classically — is ever free. All fields possess energy which curves the spacetime in which it is propagating. It is easy to see that this nonlinearity through self-gravity cannot be ignored in the mesoscopic regime as we approach Planck scales.

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These two features are also conceptually distinct. The first feature is related to how the (effective) quantum geometry affects the matter while the second feature is related to how matter at Planck scales modifies the geometry. Nevertheless, consistency demands that we should arrive at the fundamentally same description from either direction. I will show that this is indeed what happens; both features lead us to an effective quantum (corrected) geometry which exhibits a zero-point-length in the spacetime. Surprisingly, the principle of equivalence plays an interesting and subtle role in this description.

Consider a scalar field of mass m which is propagating in a space(time) with metric g_{ik} and is treated within the context of quantum field theory in curved spacetime. I want to work with a descriptor of the dynamics of this field which is robust enough to survive (and be useful) at mesoscopic scales. The propagator for the field is a good choice for such a description. All the physics of the scalar field is contained in the standard Feynman propagator $G_{std}(x_1, x_2)$, or equivalently in $\mathcal{G} \equiv mG_{\rm std}$ which will turn out to be simpler to handle algebraically. There are three equivalent ways of defining this propagator *without using the notion of a local quantum field operator.* The first definition of the (Euclidean) propagator[1] is:

$$\mathcal{G}_{\rm std}(x_1, x_2; m) \equiv mG_{std}(x_1, x_2; m^2) = \int_0^\infty m\ ds\ e^{-m^2 s} K_{std}(x_1, x_2; s) \qquad (1)$$

where $K_{\rm std}$ is the standard, zero-mass, Schwinger (heat) kernel given by $K_{\rm std}(x_1, x_2; s) \equiv \langle x_1 | e^{s\Box_g} | x_2\rangle$. Here $\Box_g$ is the Laplacian in the background space(time). The heat kernel is a purely geometric object, entirely determined by the background geometry; all the information about the scalar field is contained in the single parameter m. The second definition of the propagator is based on the path integral sum:

$$\mathcal{G}_{\rm std}(x_1, x_2; m) = \sum_{\text{paths } \sigma} \exp -m\sigma(x_1, x_2) \qquad (2)$$

where $\sigma(x_1, x_2)$ is the length of the path connecting the two events x_1, x_2 and the sum is over all paths connecting these two events. This path integral can be defined in the lattice and computed — with suitable measure — in the limit of zero lattice spacing [1, 2]. The third definition is a variant of this, obtained by converting the path integral to an *ordinary* integral. To do this, I will introduce a Dirac delta function into the path integral sum in Eq. (2) and use the fact that both ℓ and σ are positive definite, to obtain:

$$\mathcal{G}_{\rm std}(x_1, x_2; m) = \int_0^\infty d\ell\ e^{-m\ell} \sum_{\text{paths } \sigma} \delta_D\left(\ell - \sigma(x_2, x_1)\right) \equiv \int_0^\infty d\ell\ e^{-m\ell} N(\ell; x_2, x_1) \qquad (3)$$

where we have defined the function $N(\ell; x_2, x_1)$ to be:

$$N(\ell; x_2, x_1) \equiv \sum_{\text{paths } \sigma} \delta_D\left(\sigma(x_2, x_1) - \ell\right) \qquad (4)$$

[1] I will work in a Euclidean space(time) for mathematical convenience and will assume that the results in spacetime arise through analytic continuation. This is *not* essential and one could have done everything in the Lorentzian spacetime itself; it just makes life easier.

The last equality in Eq. (3) describes the path integral as an *ordinary* integral with a measure $N(\ell)$ which — according to Eq. (4) — can be thought of as counting the *effective* number of paths[2] of length ℓ connecting the two events x_1 and x_2. Most of the time I will just write $N(\ell)$ without displaying the dependence on the spacetime coordinates for notational simplicity.

Before proceeding further, let me illustrate the form of $N(\ell)$ in the case of flat space. Expressing both $\mathcal{G}_{\rm std}(p,m) = m(p^2+m^2)^{-1}$ and $N(p,\ell)$ in momentum space, we immediately see that:

$$\mathcal{G}_{\rm std}(p^2,m) = mG_{\rm std}(p^2,m^2) = \frac{m}{m^2+p^2} = \int_0^\infty d\ell\; e^{-m\ell}\cos p\ell \tag{5}$$

showing that $N(p,\ell)$ in momentum space is given by the simple expression $N(p,\ell) = \cos(p\ell)$.

This description in terms of a propagator, defined by any of these three approaches, is totally adequate to handle the matter field, when it is propagating in a given curved spacetime. None of these definitions use the formalism of a local field theory and its canonical quantisation, notions which may not survive close to Planck scales; therefore the propagator provides a robust construct which we can rely on at mesoscopic scales.

In particular, we can ask: What happens to the propagator when we approach the Planck scales? Obviously, the classical geometrical description needs to be modified close to Planck scales in a manner which is at present unknown. It is, however, possible to capture the most important effects of quantum gravity by introducing a zero-point-length to the spacetime [3]. This is based on the idea that the *dominant* effect of quantum gravity at mesoscopic scales can be captured by assuming[3] that the geodesic distance $\sigma^2(x_1,x_2)$ has to be replaced by $\sigma^2(x_1,x_2)\to\sigma^2(x_1,x_2)+L^2$ where L^2 is of the order of Planck area $L_P^2 \equiv (G\hbar/c^3)$.

It is easy to see how the introduction of zero-point-length into the geometry modifies the propagator in Eq. (3). The existence of the zero-point-length requires us to change every path length ℓ to $(\ell^2+L^2)^{1/2}$. Therefore the quantum corrected propagator will be given by the last integral in Eq. (3) with this replacement. This leads to the expression for the propagator in an (effective) quantum geometry:

$$\mathcal{G}_{\rm QG}(x_1,x_2;m) = \int_0^\infty d\ell\; N(\ell;x_1,x_2)\exp\left(-m\sqrt{\ell^2+L^2}\right) \tag{6}$$

The modification $\ell\to(\ell^2+L^2)^{1/2}$ ensures that all path lengths are bounded from below by the zero-point-length.

We know that the original path integral in Eq. (3) had an equivalent description in terms of the heat kernel through Eq. (1). How does the modification in Eq. (6) translate to the relation between the heat kernel and the propagator? With some elementary algebra, involving Laplace transforms [5], one can show that Eq. (1) is now modified to:

$$\mathcal{G}_{QG}(x_1,x_2;m) = \int_0^\infty m\; ds\; e^{-m^2s-L^2/4s}K_{std}(x_1,x_2;s) \tag{7}$$

[2]Of course, the actual number of paths, of a given length connecting any two events, is either zero or infinity. But $N(\ell)$, defined as the inverse Laplace transform of $\mathcal{G}$ (see Eq. (3)), will be a finite quantity.

[3]Such an idea has been introduced and explored extensively in the past literature [3, 4] and hence I will pause to describe it here; I will just accept it as a working hypothesis and proceed further.

Recall that the leading order behaviour of the heat kernel is $K_{\rm std} \sim \exp[-\sigma^2(x_1, x_2)/4s]$ where σ^2 is the geodesic distance between the two events; so the modification in Eq. (7) amounts to the replacement $\sigma^2 \to \sigma^2 + L^2$ to the leading order. That makes perfect sense.

Again, let me illustrate both Eq. (6) and Eq. (7) — which are valid in arbitrary curved spacetime — in the context of the flat spacetime. Working in the momentum space and using the result $N(p, \ell) = \cos pl$ from Eq. (6), we get:

$$\mathcal{G}_{\rm QG}(p^2) = \int_0^\infty d\ell \; e^{-m\sqrt{L^2+\ell^2}} \; \cos(p\ell) = \frac{mL}{\sqrt{p^2+m^2}} K_1[L\sqrt{p^2+m^2}] \tag{8}$$

Similarly, using the expression for momentum space, zero-mass, kernel in flat space, $K_{std}(s;p) = \exp(-sp^2)$ in Eq. (7) we get:

$$\mathcal{G}_{\rm QG}(p^2) = \int_0^\infty ds \; m \; \exp\left[-s(p^2+m^2) - \frac{L^2}{4s}\right] = \frac{mL}{\sqrt{p^2+m^2}} \; K_1[L\sqrt{p^2+m^2}] \tag{9}$$

which is identical to Eq. (8).

I will now approach the same issue from a different direction. The path integral in Eq. (2) tells us that the amplitude is exponentially suppressed for paths longer than the Compton wavelength $\lambda_c \equiv \hbar/mc$. This is because the action for a relativistic particle of mass m gives the factor $\exp(-A/\hbar)$ with $A/\hbar = -mc\sigma/\hbar = -\sigma/\lambda_c$ where σ is the length of the path and $\lambda_c = \hbar/mc$ is the Compton wavelength of the particle. When the self-gravity of the matter field is introduced into the picture, another length scale — viz. the gravitational Schwarzschild radius $\lambda_g \equiv Gm/c^2$ — enters the fray. The self-gravity of a particle of mass m will strongly curve the spacetime at length scales comparable to λ_g. As I said before, at these length scales, we can no longer think of a 'free field' even in flat spacetime. In fact, it makes absolutely no sense to sum over paths with $\sigma \lesssim \lambda_g$ in the path integral. Just as paths with $\sigma \gtrsim \lambda_c$ are suppressed exponentially by the factor $\exp[-(\sigma/\lambda_c)]$, we should suppress exponentially[4] the paths with $\sigma \lesssim \lambda_g$ by another factor $\exp[-(\lambda_g/\sigma)]$. Therefore, a natural and minimal modification of the path integral sum in Eq. (2), which incorporates the self gravity of a particle of mass m, will lead to the propagator:

$$\mathcal{G}(x_1, x_2) \equiv \sum_{\text{paths } \sigma} \exp\left[-\frac{\sigma}{\lambda_c}\right] \; \exp\left[-\frac{\lambda}{\sigma}\right] = \sum_{\text{paths } \sigma} \exp\left[-m\left(\sigma + \frac{L^2}{\sigma}\right)\right] \tag{10}$$

where $L = \mathcal{O}(1)L_P$. So we have now arrived at another definition for the propagator which incorporates the Planck scale effects. This modification, given by Eq. (10) has a beautiful symmetry: The amplitude is invariant under the duality transformation $\sigma \to L^2/\sigma$; we will say more about it in the sequel.

Starting from the modifications of the quantum geometry and approaching the matter sector we argued that the propagator has to be modified into the form in Eq. (6) or, equivalently, to Eq. (7). On the other hand, starting from matter sector and incorporating the self gravity of a particle of mass m into the path integral propagator, we

[4] Why this factor should also be exponential, rather than of some other functional form, will become clear soon.

have arrived at the modification of the propagator in Eq. (10). Consistency demands that these two propagators should be identical.

Remarkably enough, they are! One can indeed give meaning to the path integral sum in Eq. (10) by defining it on a lattice and then taking the limit of zero lattice spacing. Such an exercise (see Ref. [1]) shows that the path integral sum in Eq. (10) *does* lead precisely to the result in Eq. (7).

This result is non-trivial and could not have been "guessed". The result also depends on the principle of equivalence in a subtle and interesting way. To see this, note that the Compton wavelength $\lambda_c = \hbar/(m_i c)$ is defined in terms of the *inertial* mass of the particle. The part of the path integral amplitude $\exp[-(\sigma/\lambda_c)]$ comes from combining special relativity with quantum theory and does not depend on the existence of gravity. On the other hand, the gravitational radius $\lambda_g \equiv Gm_g/c^2$ is defined in terms of the *gravitational* mass of the particle. These two factors are combined in the first equality of Eq. (10). But they can be expressed as in the second equality of Eq. (10) only because of the assumption $m_i = m_g$! If $m_i \neq m_g$ then we will end up with the argument of the exponential:

$$\frac{m_i\sigma}{\hbar c} + \frac{Gm_g}{c^2\sigma} = \frac{1}{\lambda_c}\left[\sigma + \left(\frac{m_g}{m_i}\right)\frac{L_P^2}{\sigma}\right] \tag{11}$$

Clearly, one cannot provide a purely geometrical interpretation for such a factor in the square bracket, occurring in a path integral. The addition of a universal zero-point-length to the spacetime — which independent of any parameters of the matter sector — will *not* be equivalent to the modification of the propagator due to its self-gravity if $m_i \neq m_g$. Just as classical gravity admits a purely geometrical description only because $m_i = m_g$, the quantum geometry allows a universal description in terms of zero-point-length only because of $m_i = m_g$. We now have principle of equivalence operating at Planck scales!

This result also tells us why the *exponential* form of the suppression $\exp[-(\lambda_g/\sigma)]$ — rather than some other functional form — in Eq. (10), for path lengths smaller than Schwarzschild radius, is uniquely selected. No other functional form will allows us to separate out to a purely geometrical factor in the form $f_1(m_i, m_g)f_2(\sigma)$ in the exponent. As a bonus, when $m_i = m_g$, we are *led to* the factor $[\sigma + (L^2/\sigma)]$, which exhibits the duality symmetry, $\sigma \to L^2/\sigma$. This is again a direct consequence of the principle of equivalence.

There is an alternate way of relating the two directions of approach we have adopted above. To do this, I begin by relating the two propagators $G_{\rm QG}$ and $G_{\rm std}$. It is straightforward to show, again using some Laplace transform tricks, that [5]

$$G_{\rm QG}(x_1, x_2; m^2) = -\frac{\partial}{\partial m^2}\int_{m^2}^{\infty} dm_0^2\ J_0\left[L\sqrt{m_0^2 - m^2}\right] G_{\rm std}(x_1, x_2; m_0^2) \tag{12}$$

This is equivalent to assuming that — close to Planck scales — there is an amplitude $\langle m|m_0\rangle$ for a system with mass m_0 to appear as a system with mass m. Such a feature can arise due to quantum fluctuations in the length scales as follows. If we put $m_0 = \lambda m$ and write $G_{\rm std}$ as a path integral sum, then Eq. (12) can be re-expressed in the form

$$G_{\rm QG}(x_1, x_2; m^2) = \int_1^{\infty} d\lambda\ \mathcal{A}(m, \lambda) \sum_{\text{paths }\sigma} e^{-m\lambda\sigma} \tag{13}$$

with

$$\mathcal{A}(m,\lambda) = -\frac{\lambda(Lm)}{\sqrt{\lambda^2-1}}\; J_1\left[mL\sqrt{\lambda^2-1}\right] \tag{14}$$

for $\lambda > 1$. (There is a Dirac delta function contribution at $\lambda = 1$ which I have not displayed.) This suggests the following interpretation: The presence of a mass m in the space(time) induces fluctuations in the lengths of the paths changing $\sigma \to \lambda\sigma$ with an amplitude $\mathcal{A}(m,\lambda)$. The correct propagator $G_{\rm QG}(m)$ has to be obtained by integrating over these fluctuations as well as the sum over paths.

These results tell us that as we approach Planck scales, fluctuations of quantum geometry and quantum fluctuations of matter merge with each other and acquire a unified description in terms of the zero-point-lengththanks to principle of equivalence.

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