The Kinetic Theory of the Mesoscopic Spacetime*

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Abstract

At the mesoscopic scales — which interpolate between the macroscopic, classical, geometry and the microscopic, quantum, structure of spacetime — one can identify the density of states of the geometry which arises from the existence of a zero-point length in the spacetime. This spacetime discreteness also associates an internal degree of freedom with each event, in the form of a fluctuating vector of constant norm. The equilibrium state, corresponding to the extremum of the total density of states of geometry plus matter, leads precisely to Einstein's equations. In fact, the field equation can now be reinterpreted as a zero-heat dissipation principle. The analysis of fluctuations around the equilibrium state (described by Einstein's equations), will provide new insights about quantum gravity.

The magic of kinetic theory: Counting the continuum: The kinetic theory of a fluid is based[1] on the distribution function $f(x^i, p_j)$, which *counts* the number of degrees of freedom $dN = f(x^i, p_j)d^3xd^3p$ per unit phase space volume d^3xd^3p (with the constraint $p^2 = m^2$ making the phase space six-dimensional). This description in terms of a distribution function f — which could equivalently be thought of as the *density of states* in phase space — is remarkable because it achieves the impossible! It allows us to use the continuum language and, at the same time, recognize the discrete nature of the fluid. The key new feature which allows this description is the "internal" variable p^μ which describes atoms/molecules with different *microscopic* momenta co-existing at the same event x^i.

In an identical manner, we can introduce a function $\rho_g(x^i, \phi_A)$ to describe the density of states of the mesoscopic spacetime. Here, ϕ_A (with $A = 1, 2, 3, ...$) denotes possible internal degrees of freedom (analogous to the momentum p_i for the distribution function for the molecules of a fluid) which exist as fluctuating *internal* variables at *each* event x^i. Their behaviour, at any event x^i, is determined by a probability distribution $P(\phi_A, x^i)$, the form of which depends on the microscopic quantum state of the spacetime.

It turns out that there is a natural way of defining $\rho_g(x^i, \phi_A)$, if we introduce discreteness into the spacetime through a zero-point length. Remarkably enough, this

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[1]The signature is $(-,+,+,+)$ and I use natural units with $c = 1, \hbar = 1$ and set $\kappa = 8\pi G = 8\pi L_P^2$ where L_P is the Planck length $(G\hbar/c^3)^{1/2} = G^{1/2}$ in natural units. Latin letters i, j etc. range over spacetime indices and the Greek letters α, β etc. range over the spatial indices. I will write x for x^i, suppressing the index, when no confusion is likely to arise.

procedure *also* identifies the internal variable ϕ_A as a constant norm four-vector n^a, which can be thought of as a microscopic, fluctuating, quantum variable at each event x^i. Once we determine the form of $\rho_g(x^i, n_a) \equiv \exp S_g$ and the corresponding density of states for matter, $\rho_m \equiv \exp S_m$, then the equilibrium state is determined by extremum of $\rho_g \rho_m = \exp[S_g + S_m]$. This leads to Einstein's equations, along with an elegant interpretation!

Spacetime events have finite areas but zero volume: Let us start by determining ρ_g. The two primitive geometrical constructs one can think of in a spacetime are the area and the volume. It is, therefore, natural to assume that the density of states of the mesoscopic spacetime, $\rho_g(\mathcal{P})$, at an event $\mathcal{P}$, should be some function F of either the area $A(\mathcal{P})$ or the volume $V(\mathcal{P})$ that we can "associate with" the event $\mathcal{P}$. Further, the total degrees of freedom, $\rho_g(\mathcal{P})\rho_g(\mathcal{Q})$, associated with two events $\mathcal{P}$ and $\mathcal{Q}$ is multiplicative while the primitive area/volume elements are additive. Hence this function F should be an exponential. In terms of area, for example, we then get

$$\ln \left\{ \begin{array}{c} \text{density of states of the} \\ \text{quantum geometry at } \mathcal{P} \end{array} \right\} \propto \left\{ \begin{array}{c} \text{area "associated with"} \\ \text{the event } \mathcal{P} \end{array} \right\} \tag{1}$$

That is, $\ln \rho_g(\mathcal{P}) \propto A(\mathcal{P})$.

We next need to give a precise meaning to the phrase, area (or volume) "associated with" the event $\mathcal{P}$. To do this, consider the Euclidean extension of a local neighbourhood around $\mathcal{P}$ and all possible geodesics emanating from $\mathcal{P}$. The surface $\mathcal{S}(\mathcal{P}, \sigma)$ formed by all the events, at a geodesic distance σ, forms a *equi-geodesic surface* around $\mathcal{P}$. Let $A(\mathcal{S})$ its area and $V(\mathcal{S})$ be the volume enclosed by it. The limiting values of $A(\mathcal{S})$ and $V(\mathcal{S})$ when $\sigma \to 0$ provide a natural definition of the area "associated with" the event $\mathcal{P}$.

In standard Riemannian geometry — which knows nothing about the discreteness of microscopic spacetime — both the area and volume will vanish in the limit of $\sigma \to 0$, as to be expected. But when we introduce the discreteness of the spacetime in terms of a zero-point length, we find that [1] the area associated with an event becomes nonzero but the volume will still remain zero. What is more, this approach will introduce an arbitrary, constant norm, vector n_a into the discussion. (Its norm is unity in the Euclidean sector and it will map to a null vector with zero norm in the Lorentzian sector.). The area "associated with" an event will still be a fluctuating, indeterminate variable depending on a quantum degree of freedom n^a. In terms of this internal, vector degree of freedom, the $\rho_g(x^i, n_a)$ is given by

$$\ln \rho_g \propto \left[1 - \frac{L_P^2}{8\pi} R_{ab}(x) n^a n^b\right] = \mu \left[1 - \frac{L_P^2}{8\pi} R_{ab} n^a n^b\right] \tag{2}$$

where μ is a dimensionless proportionality constant. The fact that the term involving R_{ab} comes with a *minus sign* in Eq. (2) is crucial for the success of our programme and we have *no* control over it!

The mesoscopic relic from the discreteness of quantum spacetime: We see that ρ_g depends on the extra, internal degree of freedom, n_a which could take all possible values (at a given x^i) except for the constraint that it has a constant norm. *This quantity is a relic of the discrete nature of the spacetime.* This is analogous to the p_j which —

as a relic of the discrete nature of the fluid — appears in the fluid distribution function $f(x^i, p_j)$ and takes all possible values (except, again, for the constant norm constraint, $p^2 = m^2$) at a given x^i. The fluctuations of n^a are governed by some probability functional $P[n^i(x), x]$, which is the probability that the quantum geometry is described by a vector field $n_a(x)$ at every x. The exact form of P, of course, is unknown at present since we do not have the full theory of quantum gravity; but, fortunately, we will need only two properties of this probability distribution $P[n^i(x), x]$ which can be derived: (i) It preserves the norm of n^i, which is unity in the Euclidean sector and zero in the Lorentzian sector; i.e $P[n^i(x), x]$ will have the form $F[n(x), x]\delta(n^2 - \epsilon)$ with $\epsilon = 1$ in the Euclidean space and zero in the Lorentzian spacetime. (ii) The average of n^a over the fluctuations (in a given quantum state of the geometry) gives,[2] in the Lorentzian spacetime, a null normal $\ell^a(x^i)$ to a patch of null surface; i.e., $\langle n^a \rangle = \ell^a(x^i)$. Of course, different quantum states of the spacetime geometry will lead to different P with different null normals $\ell_a(x^i)$ as their mean values; so, in fact, the expectation value $\langle n_a \rangle$ actually leads to the *set of all null normals* $\{\ell_a(x^i)\}$ at an event x^i when we take into account all quantum states. Similarly, $\langle n_i n_j \rangle = \ell_i \ell_j + \sigma_{ij}$ where the second term σ_{ij} represents quantum gravitational corrections to the mean value, etc. Therefore, the mean value $\langle \ln \rho_g(x^i, n_a) \rangle$, in the continuum limit, is given by:

$$\langle \ln \rho_g(x^i, n_a) \rangle \propto 1 - \frac{1}{8\pi} L_P^2 R_{ab} \ell^a \ell^b + ... \tag{3}$$

where we have not displayed terms proportional to $R_{ab}\sigma^{ab}$ which are of higher order and independent of $\ell_a(x)$.

Density of states for matter: In the continuum limit, it is straightforward to show [2] — using the concept of local Rindler horizons associated with the null surface— that the density of states for matter is given by:

$$\langle \ln \rho_m \rangle \propto S_m \propto L_P^4 T_{ab} \ell^a \ell^b = L_P^4 \mathcal{H}_m \tag{4}$$

where $\mathcal{H}_m$ can also be interpreted as the heat density contributed by matter crossing a null patch. Note that, if T^a_b is due to an ideal fluid, then $T_{ab}\ell^a\ell^b = \rho + P = Ts$ is indeed the heat (entropy) density, where the last equality follows from Gibbs-Duhem relation. What we have in Eq. (4) is a generalization of this result to *any* T^a_b as seen by a local Rindler observer very close to the horizon.

The equilibrium state of matter and geometry: Taking into account both matter and spacetime, the total number of degrees of freedom, in the continuum limit — in a state characterized by the vector field $\ell_a(x)$ — will be

$$\langle \Omega_{\rm tot} \rangle_\ell = \prod_x \langle \rho_g \rangle \langle \rho_m \rangle = \exp \sum_x \left(\langle \ln \rho_g \rangle + \langle \ln \rho_m \rangle \right) \equiv \exp[S_{\rm grav}(\ell) + S_{\rm m}(\ell)] \tag{5}$$

to the lowest order (i.e., when we ignore the fluctuations, so that $\ln\langle \rho \rangle \approx \langle \ln \rho \rangle$). The ℓ_a dependent part of the configurational entropy $S_{\rm tot} = S_{\rm grav} + S_{\rm m}$ is given by the

[2]This is analogous to the average value of the microscopic momenta of fluid particles $\langle p^\mu \rangle = P^\mu(x^i)$ giving rise to the macroscopic momentum of the fluid. Recall that the null normal ℓ_a also defines the tangent vector to the null geodesic congruence on the null surface; in this sense, it is indeed the momentum of, say, the photons traveling along the null geodesics.

functional

$$S_{\rm tot}[\ell(x)] = \int_{\mathcal{S}} d^3V_x\ \mu E^a_b \langle n_a n^b\rangle = \int_{\mathcal{S}} d^3V_x\ \mu \left(T^a_b(x) - \frac{1}{\kappa}R^a_b(x)\right)\ell_a(x)\ell^b(x) + ... \quad (6)$$

where, in the continuum limit, the sum over x is replaced by integration over the null surface $\mathcal{S}$ for which $\ell^a(x)$ is the normal, with the measure $d^3V_x = (d\lambda d^2x\sqrt{\gamma}/L_P^3)$ and the proportionality constant μ is introduced.

The gravitational field equations can be obtained by extremizing the expression for $\langle\Omega_{\rm tot}\rangle_\ell$ (or, equivalently, the configurational entropy $S_{\rm tot} = S_{\rm grav} + S_{\rm m}$) over ℓ and demanding that the extremum condition holds for all ℓ_a. Since different quantum states of geometry will lead to different $\ell_a(x)$ at the same event x^i, this is equivalent to demanding the validity of the extremum condition for all quantum states of the geometry, which are relevant in the classical limit. Demanding that Eq. (6) is an extremum with respect to $\ell^a \to \ell^a + \delta\ell^a$, (subject to the constraint $\ell^2 =$ constant) for all ℓ_a leads to $R^a_b - \kappa T^a_b = f(x)\delta^a_b$. Taking the divergence of this equation and using $\nabla_a T^a_b = 0$ and $\nabla_a R^a_b = (1/2)\partial_b R$, you get $f(x) = (1/2)R+$ a constant, leading to Einstein's equations, with a cosmological constant arising as an integration constant. So, the classical limit makes perfect thermodynamic sense.

Einstein's equation interpreted as a Zero-Dissipation-Principle: What does this result actually mean? *Unlike in standard Einstein's theory we now have a simple physical interpretation for the field equations!* Remarkably enough, the quantity

$$\mathcal{H}_g \equiv -\frac{1}{8\pi L_P^2} R_{ab}\ell^a\ell^b \quad (7)$$

which determines the density of states of geometry, has an interpretation as the gravitational heat density (i.e., heating rate per unit area) of the null surface to which ℓ_a is the normal. Its integral over the null surface, Q_g, can be interpreted as the gravitational contribution to the heat content of the null surface. These results arise because the term $R_{ab}\ell^a\ell^b$ is related to the concept of "dissipation without dissipation" [3] of the null surfaces.

Let me explain briefly how this interesting interpretation comes about. Construct the standard description of a null surface by introducing the complementary null vector k^a (with $k^a\ell_a = -1$) and defining the 2-metric on the cross-section of the null surface by $q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b$. Define the expansion $\theta \equiv \nabla_a \ell^a$ and shear $\sigma_{ab} \equiv \theta_{ab} - (1/2)q_{ab}\theta$ of the null surface where $\theta_{ab} = q^i_a q^j_b \nabla_i \ell_j$. (It is convenient to take the null congruence to be affinely parametrized.) One can then show that [2] the integral of $R_{ab}\ell^a\ell^b$ over a null surface is

$$Q_g = -\frac{1}{8\pi L_P^2}\int \sqrt{\gamma}\, d^2x\, d\lambda\, R^a_b \ell_a \ell^b = \int \sqrt{\gamma}\, d^2x\, d\lambda\ \left[2\eta\sigma_{ab}\sigma^{ab} + \zeta\theta^2\right] \quad (8)$$

where the integrand $\mathcal{D} \equiv \left[2\eta\sigma_{ab}\sigma^{ab} + \zeta\theta^2\right]$ is the standard expression for the viscous heat generation rate of a fluid with shear and bulk viscous coefficients [4–6] defined as $\eta = 1/16\pi L_P^2, \zeta = -1/16\pi L_P^2$. So the $\mathcal{H}_g$ in Eq. (7) is just the heat density of gravity on a null surface.

Of course, we do not want every null surface to exhibit heating or dissipation! This is ensured by the presence of matter *which is needed* if $R_{ab} \neq 0$. The contribution to the heating from the microscopic degrees of freedom of the spacetime precisely cancels out the heating of any null surface by the matter, on-shell. In fact, this allows us to reinterpret the field equation, expressed as

$$-\frac{1}{8\pi L_P^2} R_{ab}\ell^a\ell^b + T_{ab}\ell^a\ell^b = \mathcal{H}_g + \mathcal{H}_m = 0 \tag{9}$$

as a "zero heat dissipation" principle.

Beyond Einstein: The spacetime fluid and its kinetics: In the kinetic theory of a normal fluid, the distribution function $f(x^a, p_i)$ — which counts the microscopic degrees of freedom — depends not only on x^a but also on the internal variable p_i, which is a fluctuating four-vector of constant norm. This internal variable is a relic of the discrete nature of the fluid, viz. the existence of atoms/molecules of matter.

Similarly, when we develop the kinetic theory of the mesoscopic spacetime, counting the corresponding mesoscopic degrees of freedom of geometry $\rho(x^i, n_a)$ (through a simple limiting procedure, associating an area with each event), we *discover* that it depends not only on x^i but also on a internal variable n_a. This internal variable, again, is a fluctuating four-vector of constant norm and is a relic of the discrete nature of the spacetime fluid.

This *discovery* of the mesoscopic spacetime degree of freedom n_a allows us to define the equilibrium state for matter and geometry, purely from combinatorics — viz., by maximizing the total degrees of freedom. The mean value of n^a maps to a null vector field ℓ^a in the semi-classical limit. Different quantum states of the geometry will lead to different ℓ^a at a given event. The demand that the extremum condition should hold for all ℓ^a is equivalent to demanding that the extremum condition holds for all quantum geometries; rather, the semi-classical spacetime arises only when this condition holds for all quantum geometries which are relevant for such a spacetime. This extremum condition, in turn, leads to Einstein's equations!

The concept of equilibrium also acquires a direct physical meaning in the classical limit, in which we identify the mean value $\langle n_a \rangle$ of the fluctuating internal variable with a null vector field $\ell_a(x)$. The equilibrium condition then reduces to the statement $\mathcal{H}_g + \mathcal{H}_m = 0$, where $\mathcal{H}_g$ is the dissipational heat density of gravity and $\mathcal{H}_m$ is the corresponding quantity for matter. Equilibrium tantamounts to zero-dissipation, as to be expected. In fact, we can show that the fluctuations away from the equilibrium are governed by the standard Boltzmann factor

$$F = \exp -\frac{1}{T_P}\left(\frac{1}{8\pi L_P^2} R^a_b(x) - T^a_b(x)\right) n_a n^b \tag{10}$$

where T_P is the Planck temperature. The extremum condition, leading to the vanishing of the argument of the exponent ensures that these degrees of freedom are not excited at the lowest order. This translates, in the classical limit, to the zero-dissipation-principle we obtained earlier.

In other words, we now have a strikingly simple physical meaning for the gravitational field equations. In the standard form, Einstein's equation $G^a_b = (8\pi L_P^2) T^a_b$, equates apples to oranges, i.e geometry with matter. In our case, we relate the geometrical heat of dissipation, $R^{ab}n_an_b$ (arising from the coupling of internal variable n_a with geometry) and heat of dissipation of matter, $T^{ab}n_an_b$ (arising from the coupling of internal variable n_a with matter).

This also suggests a possible way of understanding the question: What is the *actual mechanism* by which T^a_b generates R^a_b ? In Einstein's theory it is just a hypothesis, in the form of the field equation. In our approach, both the spacetime geometry and matter couples to n_a through the terms $R^a_bn_an^b$ and $T^a_bn_an^b$ respectively, thereby leading to an effective coupling between them. The Einstein's equation is just an average, equilibrium condition and we will expect — as in any statistical system involving large number of degrees of freedom — fluctuations around this equilibrium. This opens up new, exciting vistas of exploration.

References

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