GR and Classical Mechanics: Magic?

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(**abstract**)

A new *fundamental* ingredient is introduced in the study of Asymptotically Flat Einstein-Maxwell Space-Times, namely the change of coordinate systems from the standard ones constructed from the infinite number of possible Bondi null-surfaces to those based on the four complex-parameter set, z^a, of Asymptotically Shear-Free (ASF) null surfaces. ASF coordinate systems are determined by "world-lines" in the parameter space, $z^a = \xi^a(\tau)$. Setting a Weyl tensor component, **defined** as the *complex-mass-dipole,* to zero, a *unique complex center of mass/charge 'world-line'* is obtained. From this line and Bianchi identities, much of classical mechanics is directly obtained: spin, orbital angular momentum, kinematic momentum, angular-momentum conservation, energy-momentum conservation, Newton's 2nd law with Abraham-Lorentz-Dirac radiation reaction, Rocket force and Dirac g-factor.

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I. One of the most profound developments in General Relativity was Bondi's idea of using null surfaces to construct coordinates for use in studying and solving the Einstein Equations. A particular class of such systems (with its associated null-tetrad), now referred to as a Bondi system[1] , has been systematically used, since the 1960's, to study and solve the Einstein (Einstein-Maxwell) equations in the (future null direction) asymptotic region. The far limit of this region, defined via conformal compactification[2] I^+, is a null three-surface, usually coordinatized by the Bondi coordinates $(u,\zeta,\overline{\zeta})$. For vizualization one thinks of I^+ as a past null cone with apex at future time-infinity, with non-intersecting 'slices' or 'cuts', two-surfaces, S^2, labeled u. The *sphere* of null generator of I^+ are labeled by the complex stereographic coordinates, $(\zeta, \overline{\zeta})$,

$$\zeta = \cot\frac{\theta}{2}e^{i\phi}. \tag{1}$$

These I^+ coordinates are extended into the interior, by introducing the radial-coordinate, r, the affine parameter along the geodesic congruence normal to the slices, $u = const$. To complete a Bondi system, null tetrads, $(l, m, \bar{m}, n)$, are introduced with l^a, the tangent vector of the normal congruence, m^a and $\bar{m}^a$ the complex tangent vectors to the $u = const$ slices and n^a, the null tangents to the generators of I^+. A Bondi system is given uniquely up to the transformations, the BMS group[1]; the Lorentz group plus the supertranslations (including Poincare translations).

Almost all our discussion involves geometric quantities, e.g., the asymptotic Weyl tensor, that live and evolve on I^+, depending only on $(u,\zeta,\overline{\zeta})$. From the known asymptotic solutions of the Einstein-Maxwell equations we have the asymptotic Bianchi Identities, B.I., (Equations for the Weyl tensor), a main tool for our analysis[3] .

An important variable for the study of gravitational radiation, (radiation data), is the complex Bondi "asymptotic shear", $\sigma^0(u,\zeta,\overline{\zeta})$, with $\sigma = \sigma^0 r^{-2} + (r^{-3})$, the shear of the l^a congruence. σ^0 has the form,

$$\sigma^0 = Q(u)_{ij}Y^2_{2ij}(\zeta,\overline{\zeta}) \ + ..., \tag{2}$$

with quadrupole, $Q(u)_{ij} = Q_{(mass)ij} + i\ Q_{(spin)ij}$.

For *any* change in the slicing - *not necessarily* a BMS transform - the normal geodesic field will change as well as the σ^0. Eventually we use this freedom to construct a *non-Bondi* slicing that is *asymptotically shear-free*(ASF).[3]

The beautiful results of Bondi-Sachs[1,4] - which lie at the basis of the recent observations of gravitational waves - i.e., the gravitational energy and momentum loss theorems, found via the B.I., are expressed as derivatives of σ^0

By changing the slicing from Bondi cuts to new four-parameter, z^a, families of cuts with associated *ASF null geodesic congruences*, (fundamental to this work), we describe how to go far beyond the Bondi-Sach results. One obtains much of standard classical mechanics - including the Newton 2nd law - and, if the Einstein-Maxwell equations are considered, the Abraham-Lorentz-Dirac radiation reaction term is just sitting there - no calculations needed,.

II. Starting with a Bondi system at any point on I^+, i.e at $(u, \zeta, \overline{\zeta})$, consider its past light-cone. The *sphere* of *past null directions* is labeled by complex stereographic coordinates, $(L, \overline{L})$. If L is given for each point on I^+, i.e., $L(u, \zeta, \overline{\zeta})$, a null geodesic congruence Υ in the space-time is defined.

Much of this work is based on the following discussion[3]: If L satisfies the differential equation

$$\eth L + LL_{,u} = \sigma^0(u, \zeta, \bar{\zeta}), \tag{3}$$

$$\eth L \equiv \partial_\zeta[(1 + \zeta\bar{\zeta})L] \tag{4}$$

the congruence, Υ, is ASF[3]. The issue of finding one-parameter families of cuts so that Υ is normal to them is tricky.

If we define cut(s) by $u = G(\zeta, \overline{\zeta})$, with G a solution of

$$\eth^2 G = \sigma^0(G, \zeta, \bar{\zeta}), \tag{5}$$

it is known[3] that the general regular solution is determined by four complex parameters z^a, i.e., $u = G^*(z^a, \zeta, \overline{\zeta})$ or choosing a world-line, $z^a = \xi^a(\tau)$, in the z^a space (H-Space), the solution can be written as a one-parameter family of cuts;

$$u = G^*(\xi^a(\tau), \zeta, \overline{\zeta}) = G(\tau, \zeta, \overline{\zeta}). \tag{6}$$

With

$$L = \eth_{(\tau)} G \tag{7}$$

[$\eth_{(\tau)}$ means $\eth$ holding τ constant], by using implicit derivatives[3], one finds that L satisfies Eq.(3) and hence we have a one-parameter family of slicings whose normal congruence is ASF.

Aside: Unfortunately these cuts, $u = G(\tau, \zeta, \overline{\zeta})$, are in general complex and the analysis requires explanation. The basic idea is: *use G as complex* until the end of a calculation and then use *only the real parts* after all *differentiations* have taken place.[3,5,6]

We next need the leading Weyl and Maxwell tensor components (in NP notation),

$$\Psi_1 = \Psi_1^0(u, \zeta, \bar{\zeta})r^{-4} + 0(r^{-4}), \qquad \Psi_2 = \Psi_1^0(u, \zeta, \bar{\zeta})r^{-3} + 0(r^{-4}), \tag{8}$$

$$\phi_0 = \phi_0^0(u, \zeta, \bar{\zeta})r^{-3} + ., \quad \phi_1 = \phi_1^0(u, \zeta, \bar{\zeta})r^{-2} + ., \quad \phi_2 = \phi_2^0(u, \zeta, \bar{\zeta})r^{-1} + .. \tag{9}$$

They satisfy two asymptotic B.I. and two asymptotic Maxwell equations;

$$\dot{\Psi}_2^0 = \eth^2\overline{\sigma}^0 \; - \; \sigma^0(\bar{\sigma}^0)^{\cdot\cdot} + k\phi_2^0\overset{0}{\phi_2}, \tag{10}$$

$$\dot{\Psi}_1^0 = -\eth\Psi_2^0 + 2\sigma^0 + 2k\phi_1^0\bar{\phi}_2^0, \tag{11}$$

$$\dot{\phi}_1^0 = -\,\eth\phi_2^0, \quad \dot{\phi}_0^0 = -\eth\,\phi_1^0 + \sigma^0\phi_2^0. \tag{12}$$

The coefficients of the $l = 0$ and 1 harmonics of Ψ_2^0 are proportional to the Bondi-Sachs Energy-Momentum four-vector, (M_B, P_B^i). We ***define*** the $l = 1$

harmonic coefficient of Ψ_1^0 as proportional *to the complex mass* **dipole moment**,

$$D^i_{(complexmassD^i)} = D^i_{(mass)} + ic^{-1}J^i, \tag{13}$$

with D^i_{mass}, the mass dipole moment and J^i total angular momentum.

The $l = 1$ coefficient of ϕ_0^0 is proportional to the complex E&M dipole moment,

$$D^i_{(\text{complex}\, E\&MD^i)} = D^i_E + iD^i_M, \tag{14}$$

with (D^i_E, D^i_M) the electric and magnetic dipole moments.

We transform all the Weyl and Maxwell components and the B.I. from the Bondi system to the (ASF) system with an arbitrary world-line, $z^a = \xi^a\ (\tau)$, (a long tedious calculation[3]) - and then determine $\xi^a(\tau)$ as the complex center of mass/center of charge world-line by requiring both $D^i_{(complexmassD^i)}$ and $D^i_{(\text{complex}\, E\&MD^i)}$ to vanish. This allows us to express the Ψ_1^{0i} of the *original Bondi system* in terms of the (spatial) center of mass world-line, $\xi^i(\tau)$ and the (M_B, P^i_B) :

$$\Psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2}M_B\xi^i + i\frac{6\sqrt{2}G}{c^3}P^k_B\xi^j\epsilon_{kji} \quad + ... \tag{15}$$

Writing $\xi^i = \xi^i_R + i\xi^i_R$, using Eq.(0.13), the real and imaginary parts of Eq.(0.15) yield:

RESULT I:

$$D^i_{(mass)} = M_B\xi^i_R - c^{-1}P^k_B\xi^j_I\epsilon_{jki} + .. \tag{16}$$

$$J^i = cM_B\xi^i_I + P^k_B\xi^j_R\epsilon_{ijk} + ... \tag{17}$$

which is part of the relativistic angular momentum *tensor*, (the *relativistic* mass Dipole expression and the angular momentum; spin, $S = cM_B\xi_I$, plus orbital angular momentum, $L = r\text{x}P_B$).

Substituting Eq.(16) and (17) into the B.I. (11), yields, from the Real part, the kinematic expression for momentum with the *radiation reaction contribution* and from the Imaginary part, *exactly the Landau-Lifshitz angular momentum conservation law*,[7] with a further contribution from spin loss (new?!):

RESULT II

$$P^i_B = M_B\xi^i_R{}' - \frac{2q^2}{3c^3}\xi^i_R{}'' + ... \tag{18}$$

$$J^{*i}{}' \equiv (J^i \ + \frac{2q^2}{3c^3}\xi^i_I{}')' \ = \ \frac{2q^2}{3c^3}(\xi^{j\prime}_R\xi^{k\prime\prime}_R + \xi^{k\prime}_I\xi^{k\prime\prime}_I)\epsilon_{kji} + ... \tag{19}$$

Prime denotes $\tau-$derivatives. Dots indicate higher order terms. We have used Eq.(12) and equality of the center of mass with center of charge.

The $l = 0$ harmonic term of B.I, Eq.(10), is (Bondi's) mass loss expression augmented by further losses:

RESULT III

$$M_B' = -\frac{G}{5c^7}(Q_{Mass}^{jk'''}Q_{Mass}^{jk'''} + Q_{Spin}^{jk'''}Q_{Spin}^{jk'''}) - \frac{4q^2}{3c^5}(\xi_R^{i''}\xi_R^{i''} + \xi_I^{i''}\xi_I^{i''})$$
$$- \frac{4}{45c^7}(Q_E^{jk'''}Q_E^{jk'''} + Q_M^{jk'''}Q_M^{jk'''}). \tag{20}$$

The first term is the standard Bondi quadrupole mass loss (including the spin-quadrupole contribution to the loss - new?!), the second term and third terms are the standard E&M dipole and quadrupole energy loss, which arise from the Maxwell Equations.

From the $l = 1$ terms we get Sach's momentum loss**,**

$$P_B^i{}' = F_{recoil}^i\ , \tag{21}$$

where F_{recoil}^i is composed of non-linear radiation terms involving time-derivatives of the gravitational quadruple and E&M dipole and quadrupole moments - details not relevant now.

However substituting P_B^i from Eq.(18) into Eq.(21) leads to Newton's second law with the rocket term and radiation reaction:

RESULT IV

$$M_B\xi_R^i{}'' = F^i \equiv M_B'\xi_R^{i\prime} + \frac{2q^2}{3c^3}\xi_R^{i\prime\prime\prime} + F_{recoil}^i. \tag{22}$$

Note: The imaginary part of the complex center of mass position, ξ_I^i, is used to *determine* both the spin-angular momentum $S^i = mc\xi_I^i$ and the magnetic dipole moment $D_M^i = q\xi_M^i$. They, with the g-factor expression,

$$g = \frac{2mc}{q}\frac{D_M}{S} \tag{23}$$

leads to the Dirac value of g;

RESULT V

$$g = 2. \tag{24}$$

Summary Transitioning from Bondi coordinates to those based on the four complex parameter set, z^a, of ASF null surfaces, *using only standard GR*, we obtained, from the Asymptotic B.I. *, without any related space-time,* our dynamical results. The center of mass world-line is in the *parameter space* of asymptotically shear-free null surfaces. If this were in flat space, many of the shear-free surfaces would be just light-cones and the parameters would be just the space-time points at their apex - but not so in the curved space-time case. The backward directed rays do not focus to a point - though they seem to.

The enigma is what do these results mean? Are they meaningful? Is there anything deep here? Is it just a coincidence? How could it be a coincidence with all numerical factors correct. How did the radiation reaction force appear with virtually no calculations, simply resting in a B.I. - normally a lengthy calculation. If it is meaningful, are the very small new predictions measurable? As they are part of GR, how do they impact Quantum Gravity? Compounding the mystery is the fact that the z^a parameters are coordinates of H-space[8], a complex self-dual metric space with vanishing Ricci tensor.

We do not know the answers - but do have clues to be worked out.

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