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On the Physical Interpretation of Asymptotically Flat Gravitational Fields

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Abstract

A problem in general relativity is how to extract physical information from solutions to the Einstein equations. Most often information is found from special conditions, e.g., special vector fields, symmetries or approximate symmetries. Our concern is with asymptotically flat space-times with approximate symmetry: the BMS group. For these spaces the Bondi four-momentum vector and its evolution, found at infinity, describes the total energy-momentum and the energy-momentum radiated. By generalizing the simple idea of the transformation of (electromagnetic) dipoles under a translation, we define (analogous to center of charge) the center of mass for asymptotically flat Einstein-Maxwell fields. This gives kinematical meaning to the Bondi four-momentum, i.e., the four-momentum and its evolution is described in terms of a center of mass position vector, its velocity and spin-vector. From dynamical arguments, a unique (for our approximation) total angular momentum and evolution equation in the form of a conservation law is found.

I. Introduction

The issue of giving physical meaning or interpretation[1] to any given solution or class of solutions of the Einstein equations is essentially an unsolved problem. The origin of this difficulty lies in the arbitrariness in the choice of coordinates. In the general case, aside from a few interpretative techniques, e.g., the use of geodesic deviation, there appear to be no widely applicable techniques. The usual procedures depend on the existence of special conditions or situations, e.g., the existence of special vector fields or congruences, algebraically special metrics, available flat-space approximations, symmetries or approximate symmetries. Probably the largest class of metrics with an approximate symmetry that has been studied is that of asymptotically flat space-times, where the approximate symmetry (acting on future null infinity) is the BMS group.[2] For such space-times one can interpret[3] certain terms in the asymptotic Weyl tensor as the total energy-momentum of the interior source and identify gravitational radiation, i.e., the Bondi energy and momentum loss, from the asymptotic Bianchi identities.

It is the purpose of this note to extend these results and identify (again at null infinity) a center of mass for the interior sources and a total angular momentum vector, with evolution equations (from the Bianchi identities) for each. Unfortunately, the technical details are long and involved so that only an outline and summary of the basic ideas can be given here. While most attacks on this problem have come from group theoretic arguments,[4–7] ours has a very different starting point. Our discussion, though far from mainstream ideas, is strictly based on observing certain properties of the Maxwell and Einstein-Maxwell equations. It does not involve new physics.

We start with the Maxwell equations in Minkowski space and the trivial observation of the transformation properties of the electric dipole moment in electrodynamics under a shift of the origin:

$$\vec{D}_E^* = \vec{D}_E - Q\vec{R}. \tag{1}$$

The obvious choice of the (time-dependent) translation, $\vec{R} = \vec{D}_E/Q$, determines the center of charge world-line, making the dipole moment vanish. Though we are dealing with *real electromagnetic fields in real Minkowski space*, this trivial observation can be formally generalized[8,9,17] to complex translations defining a complex center of charge world-line around which both the electric and magnetic dipoles vanish. Associated with this complex center of charge world-line is a null geodesic congruence in Minkowski space that is shear-free but twisting. This twisting congruence generalizes the shear-free and *twist-free* light-cone congruences emanating from real Minkowski space world-lines. The complex light-cones emanating from a complex curve in complex Minkowski space when projected into the real Minkowski space generates the *shear-free but twisting congruence.*

This observation about flat-space complex curves and light-cones can be generalized to asymptotically flat space-times where *shear-free* congruences are generalized[10] to *asymptotically* *shear-free* congruences. The result: every *regular asymptotically* *shear-free congruence* is generated by an arbitrary *complex curve* in the space of the complex Poincare translation sub-group of the complexified BMS group. One then seeks complex curves that encoded complex center of charge or center of mass world-lines around which the respective complex dipoles vanish - in perfect analogy with the flat-space Maxwell case.

For Einstein-Maxwell fields there are two different asymptotically shear-free null geodesic congruences and two complex world-lines (center of mass and center of charge): one from a complex shift of 'origin' for the Maxwell dipole fields, one from a complex shift of 'origin' for the Weyl tensor. We consider *only the case when these curves coincide.*[11]

This unique curve enters the Weyl tensor from its construction giving the Bondi energy-momentum vector kinematical meaning, i.e., the momentum becomes $P = Mv + ...$, with v being the velocity vector of the real part of the world-line. The Bondi momentum loss equation then has the form of Newton's 2^{nd} law, i.e., equations of motion for the center of mass motion. The imaginary part of the world-line becomes the spin angular-momentum. It, with an orbital and precessional contribution, becomes the total angular momentum, satisfying, via the Bianchi identities, a conservation law. There are many checks and cross-checks with already known physics, e.g., radiation reaction terms, electromagnetic angular momentum loss, that support our interpretations.

The calculations, (approximations, keeping terms up to 2^{nd} order in deviations from Reissner-Nordstrom) are only summarized here. The spherical harmonic expansions[12] (with Clebsch-Gordon expansions), are kept to $l = (0, 1, 2)$ harmonics. We discuss only the vacuum case and relegate the Einstein-Maxwell results to the conclusion.

II. Some Details

The discussion is divided into two parts: first we make several observations concerning well known dynamical equations (the asymptotic Bianchi) and then we introduce, at null infinity, the complex asymptotic dipole, its transformation law and associated complex center of mass.

1. The early part

Starting with future null infinity,[13] $\mathfrak{I}^+$, (with Bondi coordinates $(u, \zeta, \overline{\zeta})$, u the Bondi time and $(\zeta, \overline{\zeta})$ the complex stereographic coordinates labeling the generators), our relevant asymptotic Bianchi identities 'living' on $\mathfrak{I}^+$ are[14,15]

$$(\psi_1^0)^{\cdot} = -\eth\psi_2^0 + 2\sigma\eth(\overline{\sigma})^{\cdot} \tag{2}$$

$$(\psi_2^0)^{\cdot} = -\eth^2(\overline{\sigma})^{\cdot} - \sigma(\overline{\sigma})^{\cdot\cdot} \tag{3}$$

$$\psi_2^0 - \overline{\psi}_2^0 = \overline{\eth}^2\sigma - \eth^2\overline{\sigma} + (\sigma)^{\cdot}\overline{\sigma} - (\overline{\sigma})^{\cdot}\sigma, \tag{4}$$

where $(\psi_1^0,\, \psi_2^0)$ are asymptotic components of the Weyl tensor and $\sigma(u,\zeta,\overline{\zeta})$ is the (arbitrary) Bondi shear

$$\sigma = 24\xi^{ij}(u)Y_{2ij}^2 + ... \tag{5}$$

Introducing the mass aspect, Ψ, these equations are rewritten as

$$\begin{aligned}
(\psi_1^0)^{\cdot} &= -\eth\Psi + \eth\sigma(\overline{\sigma})^{\cdot} + 3\sigma\eth(\overline{\sigma})^{\cdot} + \eth^3\overline{\sigma} &(6)\\
\Psi^{\cdot} &= \sigma^{\cdot}\overline{\sigma}^{\cdot} &(7)\\
\Psi &= \overline{\Psi} = \psi_2^0 + \eth^2\overline{\sigma} + \sigma(\overline{\sigma})^{\cdot}. &(8)
\end{aligned}$$

Since the Bondi mass and 3-momentum are identified with the $l = (0,1)$ harmonic components of Ψ, Eq.(7) is the Bondi mass/momentum loss equation. (To agree with standard notation for retarded time we introduce $w = \sqrt{2}uc^{-1}$, so that $(^{\cdot}) = \sqrt{2}(')c^{-1}$.)

Expanding our variables in spin-s harmonics in the form[15]

$$\begin{aligned}
\psi_1^0 &= \psi_{1i}^0 Y_{1i}^1 + \psi_{1ij}^0 Y_{2ij}^1 + ... &(9)\\
\Psi &= \overline{\Psi} = -\frac{2\sqrt{2}G}{c^2}M - \frac{6G}{c^3}P^i Y_{1i}^0 + \Psi^{ij}Y_{2ij}^0 + .. &(10)
\end{aligned}$$

we see from Eq.(8) that both the Bondi mass and momentum, M and P^i, are real.

Using the reality of P^k, the real and imaginary parts of the $l = 1$ harmonic component of Eq.(6) are

$$\begin{aligned}
\frac{\sqrt{2}}{2c}\psi_{1k}^{0\prime}|_R &= -\frac{6G}{c^3}P^k - \frac{2\cdot(12)^3}{5c}(\xi_R^{mj}{}'\xi_I^{lm} - \xi_I^{mj}{}'\xi_R^{lm})\epsilon_{ljk}, &(11)\\
\frac{\sqrt{2}}{2c}\psi_{1k}^{0\prime}|_I &= \frac{2\cdot(12)^3}{5c}(\xi_R^{mj}{}'\xi_R^{lm} + \xi_I^{mj}{}'\xi_I^{lm})\epsilon_{ljk}, &(12)
\end{aligned}$$

where we have introduced the real and imaginary parts of ξ^{ij} :

$$\begin{aligned}
\xi^{ij} &= \xi_R^{ij} + i\xi_I^{ij},\\
\xi^{ij\prime} &\equiv v^{ij} = v_R^{ij} + iv_I^{ij}.
\end{aligned}$$

The imaginary part of Eq.(12) is in the form of a conservation law: the rate of change of the imaginary part of ψ_{1k}^0 equals a flux. This equation is identified as the conservation of angular momentum.

The real part determines the P^k in terms of the derivative of $\psi_{1k}^0|_R$.

2. The Transformation

The idea is to imitate the transformation, Eq.(1), at $\mathfrak{I}^+$, where the full ψ_{1k}^0, defined to be proportional to the asymptotic complex dipole moment, is the quantity to be transformed to zero. In the calculation, many of the expressions are long: the non-linear details are hidden in known expressions, N^i, N^{ij}, G^i, K^i, Π^i.

Using the arbitrary world-line associated with an asymptotically shear-free null geodesic congruence, we have the approximate transformation law[10,11] of the complex dipole:

$$\psi_{1k}^{0*} = (\psi_1^0 - 3L\psi_2^0 + ...)|_k.$$

The stereographic angle at $\mathfrak{I}^+$, $L(u,\zeta,\overline{\zeta})$, that defines the direction of the shear-free, twisting congruence, is given by[11]

$$L(u,\zeta,\overline{\zeta}) = (\xi^i(u) + N^i)Y_{1i}^1 - (6\xi^{ij}(u) - N^{ij})Y_{2ij}^1 + ...$$

where $\xi^i(u)$ is the spatial part of the world-line. The complex center of mass is then determined (analogous to setting $D^* = 0$ in Eq.(1)) by setting $\psi_{1k}^{0*} = 0$, yielding

$$\psi_{1k}^0 = (3L\psi_2^0 + ...)|_k. \tag{13}$$

After expansion, (13) becomes

$$\psi_{1i}^0 = -\frac{6\sqrt{2}G}{c^2}M[\xi^i(w) + i\frac{1}{2}\epsilon_{kji}v^k\xi^j] + G^i. \tag{14}$$

The imaginary part yields

$$\psi_{1i}^0|_I \equiv -\frac{6\sqrt{2}G}{c^3}J^i \tag{15}$$

$$J^k = Mc\xi_I^k + M(\xi_R^i v_R^j - \xi_I^i v_I^j)\epsilon_{ijk} + K^k. \tag{16}$$

while the real part, using Eq.(11), determines the *kinematical expression* for the 3-momentum:

$$P^k = Mv_R^k + \frac{M_0}{c}(v_R^i v_I^j - \xi_I^i v_R^{j\prime} - (\xi_R^i v_I^j)')\epsilon_{ijk} + \Pi^k. \tag{17}$$

The two primary results, Eqs.(17) and (16), will be discussed in the conclusions.

Returning to (7), the $l = (0,1)$ components are

$$M' = -\frac{288c}{5G}(v_R^{ij}v_R^{ij} + v_I^{ij}v_I^{ij}) \tag{18}$$

$$P^{k\prime} = F^k \equiv \frac{192c^2}{5G}(v_I^{lj}v_R^{ij} - v_R^{lj}v_I^{ij})\epsilon_{ilk}, \tag{19}$$

the standard Bondi energy-momentum loss equations if ξ^{ij} is identified with the quadrupoles by

$$\xi^{ij} = \frac{G}{12\sqrt{2}c^4}(Q_{Mass}^{ij\prime\prime} + iQ_{Spin}^{ij\prime\prime}).$$

III. Conclusion

Our method for extracting physical content from the vacuum space-times can be applied equally well to the Einstein-Maxwell equations.[15] Our major findings are:

• The mass has a kinematical correction term, not displayed here, dependent on the variable quadrupole moment.

• From the charged Kerr-Newman metrics,[16,17,11] the imaginary part of the complex position vector is identified with spin angular momentum:

$$S^i = Mc\xi_I^i.$$

From the Maxwell fields, the magnetic moment is seen[11] to be

$$\mu^i = Q\xi_I^i.$$

with Coulomb charge Q. This leads to the Dirac value of the gyromagnetic ratio, $g = 2$. It should be noted that in the elementary particle community[18] it has been speculated that all charged elementary particles with spin have this property. How our result relates to this speculation is not clear.

• One of the strongest arguments for our interpretations comes from Eq.(12), which we identified as the conservation of total angular momentum. We have (including the Maxwell field[11])

$$\begin{aligned}
\psi_{1k}^{0\prime}|_I &= \frac{2\sqrt{2}\cdot(12)^3}{5}(\xi_R^{mj\prime}\xi_R^{lm} + \xi_I^{mj\prime}\xi_I^{lm})\epsilon_{ljk} + (E\&M\ terms)_k, \\
\psi_{1i}^{0}|_I &\equiv -\frac{6\sqrt{2}G}{c^3}J^i \\
J^k &= Mc\xi_I^k + M(\xi_R^i v_R^j - \xi_I^i v_I^j)\epsilon_{ijk} + K^k + (E\&M\ terms)_k.
\end{aligned}$$

The J^k, defined as the total angular momentum, contains the spin, $(Mc\xi_I^k)$, the orbital angular momentum $(\xi_R^i P^j \epsilon_{ijk})$ and a precession term. The flux contains three terms: the gravitational quadrupole radiation, the electromagnetic quadrupole radiation and a term arising from the electromagnetic (electric and magnetic) dipole radiation. *This latter term is identical to that calculated purely from electromagnetic theory.*[20]

•The Bondi linear three-momentum, with a Maxwell field, is expressed in kinematical variables:

$$P^k = Mv_R^k - \frac{2Q^2}{3c^3}v_R^{k\prime} + \frac{M_0}{c}v_R^i v_I^j \epsilon_{ijk} + ... \tag{20}$$

the second and third terms being the classical radiation reaction term[21,20] and the spin-velocity coupling term.[19] There is no model building and no infinities to subtract.

• From the Bondi mass loss equation, in addition to identifying the gravitational quadrupole, the electromagnetic dipole and quadrupole energy losses and identifications agree exactly with the classical Maxwell results.

• By substituting the kinematic expression for the momentum, i.e., $P^k = Mv_R^k - \frac{2Q^2}{3c^3}v_R^{k\prime} + \frac{M_0}{c}v_R^i v_I^j \epsilon_{ijk} + ...$ into the Bondi momentum loss, Eq.(19), we obtain the equations of motion for our world-line. In a sense we "derive" Newton's 2^{nd} law, $Mv' = F$, where the force is a combination of electromagnetic radiation reaction,[21,20] gravitational radiation reaction and a "rocket" recoil force from the electromagnetic and gravitational momentum loss.

• Finally, it has been shown[11] that each of the quantities that were identified as physical variables, transformed properly under the Lorentz group, i.e., as Lorentzian tensorial objects.[22]

We have thus obtained the full dynamics of a radiating, isolated gravitating-electromagnetic source.

Concluding, we point out that there are unfamiliar terms in our equations that could be interpreted as predictions of our theoretical construct. How to measure them is not clear.

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IV. Research Bibliography

[1]H. Weyl, *Space-Time-Matter,* Methuen, London , (1922).

[2]R. Sachs, Proc. Roy. Soc. **270**, 103, (1962).

[3]H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, Proc. R. Soc., London Ser. A **269**, (1962) 21.

[4]L. B. Szabados, "Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article", in Living Reviews in Relativity, June 2005.

[5]Adam D. Helfer, Gen. Relativity & Gravitation, **39**, 2125-2147, (2007).

[6]Dray, T., and Streubel, M., Class. Quantum Grav., **1**, 15-26, (1984).

[7]Ashtekar, A., and Winicour, J., J. Math. Phys., **23**, 2410-2417, (1982).

[8]B. Aronson and E. T. Newman, J. Math. Phys. **13**, 1847-51, (1972).

[9]Ezra T. Newman, Class. Quantum Grav **21** 3197-3221,(2004).

[10]C. Kozameh and E.T. Newman, Class. Quantum Grav. **22,** 4659-4665, (2005).

[11]C. Kozameh, E.T. Newman and Gilberto Silva-Ortigoza, "*On Extracting Physical Content from Asymptotically Flat Space-Time Metrics*" submitted Class.Quantum.Grav., **(2008).**

[12]E.T. Newman and G. Silva-Ortigoza, Class.Quantum Grav., **23**, 497-509, (2006).

[13]R. Penrose, Phys. Rev. Ltt. **10**, 66, (1963).

[14]Ezra T. Newman and Roger Penrose, J.Math.Phys, **3,** 566–768, (1962).

[15]E.T. Newman, K.P.Tod, in *General Relativity and Gravitation*, Vol **2**, Alan Held, (ed.), Plenum (1980).

[16]E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, R. Torrence, J.Math.Phys.**6,** 918 (1965).

[17]E. T. Newman, Phys. Rev. D, **65**, 104005, (2002).

[18]M. Rivas, J.M. Aguirregabiria, A. Hernandez, Phys. Letters.A, **257**, 21-25, (1999).

[19]M. Mathisson, Proc.Camb.Phil.Soc., **38**,40-60, (1942).

[20]Landau and E. M. Lifschitz, *Classical Theory of Fields,* Addison-Wesley, Reading , Mass., (1962).

[21] W. Thirring, *Principles of Quantum Electrodynamics (p.25),* Academic Press, NY and London, (1958).

[22] A. Held, E. T. Newman and R. Posadas, J. Math. Phys. **11**, 3145 (1970).