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Non-perturbative de Sitter vacua via α' corrections

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Abstract

The higher-derivative α' corrections consistent with $O(d, d)$ duality invariance can be completely classified for cosmological, purely time-dependent backgrounds. This result is used to show that there are duality invariant theories featuring de Sitter vacua as solutions that are non-perturbative in α', thus suggesting that classical string theory may realize de Sitter solutions in an unexpected fashion.

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Since the discovery in the late 1990s that the Universe is presently undergoing accelerated expansion it has been a challenge for fundamental theories such as string theory to provide a natural explanation for this phenomenon. The arguably most convincing explanation is a positive cosmological constant leading to approximate de Sitter spacetimes. Accordingly, the challenge has been to find de Sitter solutions in string theory. But despite more than two decades of efforts not a single uncontested construction of de Sitter vacua in string theory has emerged. It has become a question of principle whether string theory permits de Sitter solutions. Recently, this state of affairs has led some researchers to conjecture that there are no de Sitter vacua in string theory and that, if string theory is to remain tenable, one has to search for other explanations of the accelerated expansion of the Universe [1]. In this essay we will use an admittedly simplified setup to argue that de Sitter vacua may be realized in classical string theory as solutions that are non-perturbative in the (inverse) string tension α'.

Classical string theory can be described by two-derivative supergravity theories augmented by an infinite number of higher-derivative corrections organized by the dimensionful parameter α'. Such corrections have been determined to a few orders in α' in the 1980s [2–4], but a complete determination or classification in general dimensions is certainly out of reach. While there has been intriguing progress in recent years encoding α' corrections in duality covariant formulations [5–8] even here a complete description of higher-derivative corrections remains out of reach. Part of the complication arises because in terms of supergravity field variables the duality transformations themselves acquire derivative corrections making them extremely unwieldy. Alternatively, when formulated in terms of duality covariant 'stringy' field variables, the gauge transformations acquire α' corrections, thereby suggesting a novel kind of geometry that, however, has only been partially developed.

Had physicists classified all the α' corrections of general $D = d + 1$ dimensional theories compatible with duality we would then explore the general cosmological solutions by using a suitable time dependent ansatz. While for now this straightforward route is blocked, it turns out that for purely time dependent cosmological solutions we *can* do a general analysis. It is in fact possible to classify all the duality invariant corrections relevant to these backgrounds. Here the duality group is $O(d, d, \mathbb{R})$ [9] and it efficiently constrains the possible corrections because, happily, the use of field redefinitions allows us to work with the simple un-corrected duality transformations of the two-derivative theory. In this way we circumvent the hard classification problem of the general spacetime theory while getting directly the most general interactions that such a classification would yield when applied to cosmology.

In the remainder of this essay we will introduce the $O(d, d, \mathbb{R})$ duality symmetry, state the result of the classification of higher-derivative cosmological interactions, and use this result to derive the most general cosmological equations for a single scale factor to all orders in α'. Equipped with these equations we show how to construct non-perturbative de Sitter solutions.

We start from the $D = d + 1$ dimensional spacetime action of closed string theory for the universal massless fields, the metric $g_{\mu\nu}$, the antisymmetric b-field $b_{\mu\nu}$, and the scalar dilaton ϕ,

$$I \;=\; \int d^Dx\sqrt{-g}\, e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \frac{1}{12}H^2 + \frac{1}{4}\alpha'\Big(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \cdots\Big)\Big)\,, \tag{1}$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}$. Here we have schematically included the first α' corrections (with the coefficient for bosonic string theory). Let us now drop the dependence on all spatial coordinates, leaving only the dependence on time t, i.e., we set $\partial_i = 0$, where $x^\mu = (t, x^i)$, $i = 1, \ldots, d$. This is the appropriate truncation for cosmology, where the non-trivial dynamics resides only in the time dependence. For the metric, antisymmetric tensor, and dilaton we then set:

$$g_{\mu\nu} \;=\; \begin{pmatrix} -n^2(t) & 0 \\ 0 & g_{ij}(t) \end{pmatrix} , \qquad b_{\mu\nu} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & b_{ij}(t) \end{pmatrix} , \qquad \phi \;=\; \phi(t) \, . \tag{2}$$

Performing this reduction in (1) to zeroth order in α' one obtains the $O(d, d, \mathbb{R})$ invariant action [10]:

$$I_0 \;\equiv\; \int dt \, e^{-\Phi} \, \frac{1}{n} \Big(-\dot{\Phi}^{\,2} - \frac{1}{8} \text{tr}\big(\dot{\mathcal{S}}^2\big) \Big) , \tag{3}$$

where the 'generalized metric' $\mathcal{S}$ is a $2d \times 2d$ matrix constructed in terms of $g_{ij}(t)$ and $b_{ij}(t)$:

$$\mathcal{S} \;\equiv\; \begin{pmatrix} bg^{-1} & g - bg^{-1}b \\ g^{-1} & -g^{-1}b \end{pmatrix} , \tag{4}$$

and the $O(d, d)$ invariant dilaton Φ is constructed in terms of the scalar dilaton and the metric determinant:

$$e^{-\Phi} \;\equiv\; \sqrt{\det g_{ij}} \, e^{-2\phi} \, . \tag{5}$$

The matrix $\mathcal{S}$ is $O(d, d, \mathbb{R})$ valued and satisfies $\mathcal{S}^2 = \mathbf{1}$. The action is manifestly invariant under $O(d, d, \mathbb{R})$ transformations:

$$\mathcal{S} \;\to\; \mathcal{S}' \;=\; h\,\mathcal{S}\,h^{-1} , \qquad \text{with} \qquad h\,\eta\,h^t \;=\; \eta \, , \qquad \eta \;=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} . \tag{6}$$

Here η is the $O(d, d, \mathbb{R})$ invariant metric. These transformations include, for vanishing b-field, the duality that sends the metric g to its inverse g^{-1}.

Let us next include α' corrections, building on the seminal work of Meissner [11] and our own subsequent improvements [12, 13]. In reference [13] we use $\mathcal{S}^2 = \mathbf{1}$ and the most general duality-covariant redefinitions to classify the higher-derivative corrections of the two-derivative action I_0. Interestingly, no corrections involve the dilaton nontrivially and *all* corrections can be written in terms of $\dot{\mathcal{S}}$, the first order time derivative of the generalized metric. We prove that the general higher-derivative term is constructed as the trace of even powers of $\dot{\mathcal{S}}$. Multiple traces are allowed but no factors $\text{tr}(\dot{\mathcal{S}}^2)$ need to be included. Thus, the most general duality invariant α' corrected action reads

$$I \;\equiv\; \int dt \; e^{-\Phi} \Big(-\dot{\Phi}^2 \;+\; \sum_{k=1}^{\infty} (\alpha')^{k-1} c_k \, \text{tr}(\dot{\mathcal{S}}^{2k}) \;+\; \text{multi-traces} \Big) , \tag{7}$$

where we included the lowest order term in the sum, with $c_1 = -\frac{1}{8}$, and picked a gauge with lapse function $n = 1$. At this stage the c_k with $k \geq 2$ are free coefficients, not determined by duality invariance. The claim is that for any string theory there are coefficients c_k (as well as coefficients for the multi-trace terms) so that (7) encodes the complete α' corrections for cosmological backgrounds. Ignoring for simplicity the multi-trace terms, the equations of motion for Φ, $\mathcal{S}$, and lapse function n are given by

$$
\begin{aligned}
0 &= 2\ddot{\Phi} - \dot{\Phi}^2 - \sum_{k=1}^{\infty} (\alpha')^{k-1} c_k \, \mathrm{tr}\big(\dot{\mathcal{S}}^{2k}\big) \, , \\
0 &= \sum_{k=1}^{\infty} (\alpha')^{k-1} k \, c_k \Big(\frac{d}{dt} \dot{\mathcal{S}}^{2k-1} - \dot{\Phi} \, \dot{\mathcal{S}}^{2k-1} + \mathcal{S} \dot{\mathcal{S}}^{2k} \Big) \, , \\
0 &= \dot{\Phi}^2 - \sum_{k=1}^{\infty} (\alpha')^{k-1} (2k-1) c_k \, \mathrm{tr}\big(\dot{\mathcal{S}}^{2k}\big) \, .
\end{aligned}
\tag{8}
$$

We now specialize to the class of Friedmann-Lemaitre-Robertson-Walker (FLRW) backgrounds with curvature $k = 0$ and vanishing b-field. The spacetime metric is then:

$$
ds^2 = -dt^2 + a^2(t) \, d\mathbf{x}^2 \, , \tag{9}
$$

where $d\mathbf{x}^2$ denotes the Euclidean spatial metric and $a(t)$ is the scale factor. For this metric the generalized metric takes the form

$$
\mathcal{S}(t) \;=\; \begin{pmatrix} 0 & a^2(t) \\ a^{-2}(t) & 0 \end{pmatrix} \, . \tag{10}
$$

Let us now evaluate the equations of motion (8) for this ansatz in order to determine the α' corrected Friedmann equations. It follows from the duality invariance (6) that these equations are invariant under the 'scale factor duality' $a \to a^{-1}$. Let us emphasize that we have in fact obtained the most general equations, despite (8) being derived from the action without multi-trace contributions. The reason is that for the ansatz (10) the multi-trace terms give the same structural contributions to the equations of motion as the single-trace terms, hence resulting only in a 'renormalization' of the free parameters c_k. The generalized Friedmann equations (8) now become

$$
\begin{aligned}
\ddot{\Phi} + \tfrac{1}{2} H f(H) &= 0 \, , \\
\frac{d}{dt}\big(e^{-\Phi} f(H) \big) &= 0 \, , \\
\dot{\Phi}^2 + g(H) &= 0 \, ,
\end{aligned}
\tag{11}
$$

where $H(t)$ is the Hubble parameter:

$$
H(t) \;=\; \frac{\dot{a}(t)}{a(t)} \, , \tag{12}
$$

and f and g are the following functions of H:

$$
\begin{aligned}
f(H) &\equiv d \sum_{k=1}^{\infty} (-\alpha')^{k-1} \, 2^{2(k+1)} \, k \, c_k \, H^{2k-1} &&= -2d \, H + \cdots \, , \\
g(H) &\equiv d \sum_{k=1}^{\infty} (-\alpha')^{k-1} 2^{2k+1} (2k-1) c_k \, H^{2k} &&= -d \, H^2 + \cdots \, ,
\end{aligned}
\tag{13}
$$

where we used $c_1 = -\frac{1}{8}$. The functions $f(H)$ and $g(H)$ are closely related and satisfy

$$g'(H) \;=\; Hf'(H)\;, \tag{14}$$

a fact that follows from one-dimensional reparametrization invariance. One may integrate this equation to express g in terms of f,

$$g(H) \;=\; Hf(H) - \int_0^H f(H')dH'\;, \tag{15}$$

for which $g(0) = 0$, in agreement with (13).

We now ask whether equations (11) permit de Sitter solutions of the form

$$ds^2 = -dt^2 + e^{2H_0 t}d\mathbf{x}^2\;, \tag{16}$$

with constant Hubble parameter $H(t) = H_0$ that in absence of matter is related to the cosmological constant by $\Lambda = 3H_0^2/c^2$. Let us first inspect the equations truncated to zeroth order in α'. Since then $f(H) \sim H_0$ the second equation of (11) implies that Φ is constant. The first equation then implies $H_0^2 = 0$, leading to $H_0 = 0$ and flat space. Unsurprisingly, the low-energy action (1) to zeroth order in α' thus does not permit de Sitter vacua. Nor do the equations truncated to first order in α' allow for de Sitter vacua. A non-perturbative de Sitter solution of (11) with $H_0 \neq 0$ requires, by the second equation, a constant Φ, so that with the first and third equation we infer $f(H_0) = g(H_0) = 0$. A de Sitter solution exists provided f and g share a common non-vanishing zero.

To see that these conditions can be met assume that f is given by the sine function,

$$f(H) \;\equiv\; -\frac{2d}{\sqrt{\alpha'}}\sin\left(\sqrt{\alpha'}\,H\right) \;=\; -2d\sum_{k=1}^{\infty}(-\alpha')^{k-1}\frac{1}{(2k-1)!}H^{2k-1}\;. \tag{17}$$

From the definition (13) it is evident that there are coefficients $c_{k\geq 2}$ so that this holds. This function has non-vanishing zeroes for $\sqrt{\alpha'}\,H_0 = n\,\pi$, $n \in \mathbb{Z}$. The zeroes associated with even, positive n are particularly interesting:

$$\sqrt{\alpha'}\,H_0 \;=\; 2\pi\,,\; 4\pi\,,\; \ldots\,, \tag{18}$$

because these are zeroes of g too. Indeed, with (15) and taking $\sqrt{\alpha'}H_0 = 2\pi n$ we have

$$g(H_0) \;=\; -\int_0^{H_0} f(H')dH' \;=\; \frac{2d}{\sqrt{\alpha'}}\int_0^{\frac{2\pi n}{\sqrt{\alpha'}}}\sin(\sqrt{\alpha'}H')dH' \;=\; \frac{2d}{\alpha'}\int_0^{2\pi n}\sin u\,du \;=\; 0\,. \tag{19}$$

Equation (18) encodes a discrete infinity of de Sitter vacua. (We chose $H_0 > 0$ because the sign is not fixed by the equations: the duality invariance $a \to a^{-1}$ takes $H \to -H$.) Note, however, that the natural values of the cosmological constant $\Lambda \sim \frac{1}{\alpha'}$ are many orders of magnitude too large to account for the observed value, which is the cosmological constant problem.

We do not mean to suggest that (17) is the function actually appearing in any string theory. It surely is not. But, as in (19), one infers that any function f will do the job if it has a non-vanishing zero H_0 so that its integral from 0 to H_0 vanishes. Alternatively, one can search for an even function $F(H) \sim -H^2$ for $H \to 0$ such that it has arguments where F and its derivative F' vanish. If so, $f(H) \equiv F'(H)$ will lead to de Sitter vacua.

Let us emphasize that the above solution is non-perturbative in α'. There are, however, polynomials $f(H)$ of degree five and higher, corresponding to only a finite number of higher-derivative terms, that satisfy the above criterion and hence lead to de Sitter vacua. This is a familiar feature of gravity with a finite number of higher-derivative terms [14]. In string theory these vacua cannot be trusted and are almost certainly an artifact of the truncation. Reliable de Sitter solutions should be non-perturbative in α', as exhibited above.

We close with a few remarks on a non-perturbative initial-value formulation of (11). Suppose we specify $\Phi(0)$ and $\dot{\Phi}(0)$ at time $t = 0$. Then the third equation of (11) (the 'Hamiltonian constraint') poses constraints on the initial value $H_0 = H(0)$ of the Hubble parameter:

$$g(H_0) \;=\; -\dot{\Phi}^2(0)\,. \tag{20}$$

The roots of this equation yield the possible values of H_0. Assuming that such a root has been picked, (11) may then be integrated to later times. Since $g(H) = -d\,H^2 + \mathcal{O}(\alpha')$, the constraint (20) may always be solved perturbatively, but non-perturbatively there may be many, few or no solutions. It has not escaped our notice that this observation has a bearing on the problem of how many vacua there are in string theory.

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