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A proof of the weak gravity conjecture

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Abstract

The weak gravity conjecture suggests that, in a self-consistent theory of quantum gravity, the strength of gravity is bounded from above by the strengths of the various gauge forces in the theory. In particular, this intriguing conjecture asserts that in a theory describing a U(1) gauge field coupled consistently to gravity, there must exist a particle whose proper mass is bounded (in Planck units) by its charge: $m/m_{\text{P}} < q$. This beautiful and remarkably compact conjecture has attracted the attention of physicists and mathematicians over the last decade. It should be emphasized, however, that despite the fact that there are numerous examples from field theory and string theory that support the conjecture, we still lack a general proof of its validity. In the present Letter we prove that the weak gravity conjecture (and, in particular, the mass-charge upper bound $m/m_{\text{P}} < q$) can be inferred directly from Bekenstein's generalized second law of thermodynamics, a law which is widely believed to reflect a fundamental aspect of the elusive theory of quantum gravity.

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It is widely believed that string theory may provide a self-consistent description of the elusive theory of quantum gravity. However, the predictive power of the theory seems to be restricted by its permissive nature [1]: there are simply too many semi-classically consistent theories of gravity that arise in string theory.

In order to distinguish the landscape of consistent theories of gravity from the swampland of low-energy effective theories which cannot be completed to a full theory of quantum gravity, Arkani-Hamed, Motl, Nicolis, and Vafa [1] have proposed the intriguing "weak gravity conjecture" as a very simple and powerful constraint on the possible gauge theories that could arise from a self-consistent quantum theory of gravity. In its simplest form, this highly interesting conjecture asserts that in a theory describing a U(1) gauge field coupled consistently to gravity, there must exist at least one state (particle) whose proper mass is bounded from above by its charge:

$$m/m_{\text{P}} < q \ , \tag{1}$$

where $m_{\text{P}} \equiv \sqrt{\hbar c/G}$ is the fundamental Planck mass [2, 3].

As originally discussed in [1], a violation of the weak gravity conjecture (1) in a theory describing a U(1) gauge field coupled to gravity would imply the absolute stability of extremal black holes in this theory. Since entropic arguments suggest that stable black-hole remnants are pathological in a quantum theory of gravity [4–6], it has been asserted in [1] that the mass-charge relation (1) is mandatory in any self-consistent theory of quantum gravity [7].

The elegant and remarkably compact weak-gravity conjecture (1) has attracted the attention of physicists and mathematicians over the last decade (see [8–14] and references therein). However, it is important to emphasize that despite the fact that there are numerous examples from field theory and string theory that support the conjecture [1, 8–14], we still lack a general proof of its validity.

One naturally wonders: where does the conjectured weak gravity bound $m/q < 1$ come from? It is not clear how to derive the suggested mass-charge relation (1) directly from microscopic quantum considerations. In particular, microscopic physics seems to afford no special status to the dimensionless mass-to-charge ratio m/q of fundamental fields.

It should be emphasized that the weak gravity conjecture is expected to characterize gauge field theories in a self-consistent quantum theory of gravity [1]. This fact suggests that a derivation of the conjectured mass-charge relation (1), which quantify the weak gravity principle, may require use of the yet unknown quantum theory of gravity. This conclusion, if true, may seem as bad news for our physical aspirations to provide a general proof of the weak gravity bound $m/q < 1$. But we need not lose heart – it is widely believed [15] that Bekenstein's generalized second law of thermodynamics [6, 16], and the closely related concept of black-hole entropy (which combines together all three fundamental constants of nature into one simple formula $S_{\rm BH} = k_{\rm B}c^3A/4G\hbar$ [6, 17]), reflect a fundamental aspect of any self-consistent quantum theory of gravity.

Interestingly, and most importantly for our analysis, it has been explicitly shown in [18, 19] that the generalized second law of thermodynamics [6] yields a fundamental lower bound on the characteristic relaxation time τ of perturbed physical systems. In particular, for a thermodynamic system of temperature T, the universal relaxation bound can be expressed by the compact time-times-temperature (TTT) relation [18, 19]

$$\tau \times T \geq 1/\pi \ . \tag{2}$$

In the context of a self-consistent quantum theory of gravity, the universal relaxation bound (2) asserts that black holes, which are known to be characterized by a well defined Bekenstein-Hawking temperature $T_{\rm BH}$ [6], must have (at least) one exponentially decaying perturbation mode $\Psi(r,t) = \psi(r)e^{-i\omega t}$ whose fundamental resonant frequency ω_0 is characterized by the relation [20, 21]

$$\Im\omega_0 \leq \pi T_{\rm BH} \ . \tag{3}$$

Interestingly, it has been proved [18, 19] that astrophysically realistic (spinning) Kerr black holes conform to the universal relaxation bound (2). In particular, the fundamental inequality (3) is saturated in the near-extremal $T_{\rm BH} \to 0$ limit of rapidly-rotating Kerr black holes [18, 19].

We shall now show explicitly that the universal relaxation bound (2), when applied to the characteristic resonant relaxation spectrum of near-extremal charged Reissner-Nordström black holes, yields in a remarkably simple way the fundamental weak gravity (mass-charge) relation (1).

The universal relaxation bound and the weak gravity conjecture.— The late-time relaxation dynamics of charged massive scalar fields in the charged Reissner-Nordström black-hole spacetime is known to be characterized by exponentially damped oscillations [22–26]. In particular, the characteristic relaxation timescale $\tau_{\rm relax}$ of the perturbed black-hole spacetime is determined by the imaginary part of the fundamental (least damped) resonant mode of the composed black-hole-field system:

$$\tau_{\rm relax} \equiv 1/\Im\omega_0 \ . \tag{4}$$

As we shall now prove, the intriguing mass-charge (weak gravity) inequality (1) can be deduced directly from the universal relaxation bound (3) as applied to the characteristic complex relaxation spectrum of near-extremal charged Reissner-Nordström black-hole spacetimes.

The physical properties of a linearized perturbation mode [27, 28]

$$\Psi(t,r,\theta,\phi)=r^{-1}e^{im\phi}S_{lm}(\theta)\psi_{lm}(r;\omega)e^{-i\omega t} \tag{5}$$

describing the dynamics of a scalar field Ψ of proper mass μ and charge coupling constant q in a Reissner-Nordström black-hole spacetime of mass M and electric charge Q are determined by the Schrödinger-like ordinary differential equation [22–26, 29]

$$\frac{d^2\psi}{dr^{*2}}+V\psi=0\ , \tag{6}$$

where the effective radial potential which characterizes the interaction of the charged massive field with the curved black-hole spacetime is given by [22–26]

$$V=V(r;M,Q,\omega,q,\mu,l)=\Big(\omega-\frac{qQ}{r}\Big)^2-\Big(1-\frac{2M}{r}+\frac{Q^2}{r^2}\Big)\Big[\mu^2+\frac{l(l+1)}{r^2}+\frac{2M}{r^3}-\frac{2Q^2}{r^4}\Big]\ . \tag{7}$$

We are interested in resonant perturbation modes of the composed black-hole-field system which are characterized by the physically motivated boundary condition $\psi(r\to r_+)\sim e^{-i(\omega-qQ/r_+)r^*}$ of purely ingoing waves at the horizon $r_+=M+(M^2-Q^2)^{1/2}$ of the black hole. In addition, the black-hole-field perturbation modes are characterized by the boundary condition $\psi(r\to\infty)\sim e^{i\sqrt{\omega^2-\mu^2}r}$ at spatial infinity [30]. The Schrödinger-like differential equation (6) with the effective radial potential (7) and the above stated physically motivated boundary conditions determine the complex resonant spectrum $\{\omega_n(M,Q,\mu,q,l)\}_{n=0}^{n=\infty}$ which characterizes the relaxation dynamics of the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system.

Interestingly, as explicitly shown in [22], the fundamental (least damped) resonances of this composed black-hole-field system can be determined *analytically* in the regime [31]

$$T_{\text{BH}}=\frac{\hbar(r_+-r_-)}{4\pi r_+^2}\to 0 \tag{8}$$

of near-extremal black holes. In particular, from equations (6)-(7) one finds the remarkably compact expression [22]

$$\Im\omega_n=2\pi T_{\text{BH}}(n+1/2+\Im\delta)\ \ \ ;\ \ \ n=0,1,2,... \tag{9}$$

for the imaginary parts of the composed black-hole-field resonances in the near-extremal regime (8), where the dimensionless field parameter δ is given by [22]

$$\delta\equiv\sqrt{M^2(q^2-\mu^2)-(l+1/2)^2}\ . \tag{10}$$

The mandatory existence of a charged particle which respects the dimensionless weak gravity relation

$$\frac{q}{\mu} > 1 \tag{11}$$

in the coupled Einstein-Maxwell theory can now be inferred by substituting $n = 0$ in the resonant relaxation spectrum (9) and requiring that $\Im\omega_0 \leq \pi T_{\text{BH}}$ [see (3)] for this fundamental black-hole resonance [32]. We have therefore proved the previously conjectured mass-charge bound (1).

Summary.— The conjectured weak gravity relation $\mu/q < 1$ [1] in theories describing a U(1) gauge field coupled consistently to gravity has been the focus of intense research during the last decade (see [8–14] and references therein). However, despite the flurry of activity in this field, the origin of this highly interesting mass-charge relation has remained somewhat mysterious. In this Letter we have explicitly shown that the generalized second law of thermodynamics [6], a physical law which is widely believed to reflect a fundamental aspect of a self-consistent quantum theory of gravity, may shed much light on the origins of this highly intriguing bound [33]. In particular, it has been explicitly proved that the dimensionless mass-charge upper bound $\mu/q < 1$ can be deduced directly from the self-consistent interplay between quantum theory, thermodynamics, and gravity.

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[3] We shall henceforth use natural units in which $G = c = \hbar = k_{\text{B}} = 1$. Note that in these units q and m have the dimensions of length^{-1}.

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[20] Note that the complex nature of the black-hole resonant frequency ω reflects the fact that the corresponding perturbation mode of the composed black-hole-field system decays over time.

[21] Here the characteristic relaxation time of the exponentially decaying black-hole perturbation mode is given by the relation $\tau_{\text{relax}} \equiv 1/\Im\omega_0$, where ω_0 is the fundamental (least damped) resonant frequency of the corresponding black-hole spacetime.

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[27] Here (t, r, θ, ϕ) are the Schwarzschild coordinates.

[28] Here $(m, l \geq |m|)$ are the azimuthal and spherical harmonic indices which characterize the exponentially damped perturbation modes of the composed black-hole-field system. For brevity, we shall henceforth omit these integer parameters.

[29] The radial coordinate r^* in (6) is related to the Schwarzschild areal coordinate r by the differential relation $dr^* = dr/(1 - 2M/r + Q^2/r^2)$.

[30] This asymptotic boundary condition corresponds to free outgoing waves (quasinormal resonances) in the large-frequency regime $\omega^2 > \mu^2$, whereas it describes spatially decaying field configurations (bound-state resonances) in the small-frequency regime $\omega^2 < \mu^2$ [22–26].

[31] Here $r_{\pm} = M \pm (M^2 - Q^2)^{1/2}$ are the (outer and inner) horizon radii of the charged black-hole spacetime.

[32] Note that the dimensionless mass-charge inequality (1) provides a necessary condition for the relation $\Im\delta = 0$. In particular, in this physical regime of the charged field parameters (with $\delta \in \mathbb{R}$), one finds a saturation of the universal relaxation bound (3) by the near-extremal ($T_{\text{BH}} \to 0$) charged black holes.

[33] Strictly speaking, we have explicitly shown in this Letter that the weak gravity relation $\mu/q < 1$ can be deduced directly from the universal relaxation bound (2) as applied to the characteristic complex relaxation spectrum of near-extremal charged Reissner-Nordström black holes. It should be remembered, however, that, as explicitly shown in [18, 19], the universal relaxation bound can be deduced directly from Bekenstein's generalized second law of thermodynamics [6]. Thus, one may regard the weak gravity relation (1) as a natural physical outcome of the generalized second law of thermodynamics.