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Gravitation, Thermodynamics, and the Bound on Viscosity

Shahar Hod

The Hadassah Institute, Jerusalem 91010, Israel

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Abstract

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence implies that small perturbations of a black hole correspond to small deviations from thermodynamic equilibrium in a dual field theory. For gauge theories with an Einstein gravity dual, the AdS/CFT correspondence predicts a universal value for the ratio of the shear viscosity to the entropy density, $\eta/s = 1/4\pi$. It was conjectured recently that all fluids conform to the lower bound $\eta/s \geq 1/4\pi$. This *conjectured* bound has been the focus of much recent attention. However, despite the flurry of research in this field we still lack a proof for the general validity of the bound. In this essay we show that this mysterious bound is actually a direct outcome of the interplay between gravity, quantum theory, and thermodynamics.

Email: shaharhod@gmail.com

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1] has yielded remarkable insights into the dynamics of strongly coupled gauge theories. According to this duality, asymptotically AdS background spacetimes with event horizons are interpreted as thermal states in dual field theories. This implies that small perturbations of a black hole or a black brane background correspond to small deviations from thermodynamic equilibrium in a dual field theory. One robust prediction of the AdS/CFT duality is a universally small ratio of the shear viscosity to the entropy density [2],

$$\frac{\eta}{s} = \frac{1}{4\pi} \ , \tag{1}$$

for all gauge theories with an Einstein gravity dual in the limit of large 't Hooft coupling. (We use natural units for which $G = c = \hbar = k_B = 1$.)

It was suggested [2] that (1) acts as a universal lower bound [the celebrated Kovtun-Starinets-Son (KSS) bound] on the ratio of the shear viscosity to the entropy density of general, possibly nonrelativistic, fluids. Currently this bound is considered a *conjecture* well supported for a certain class of field theories [3]. So far, all known materials satisfy the bound for the range of temperatures and pressures examined in the laboratory. The system coming closest to the bound is the quark-gluon plasma created at the BNL Relativistic Heavy Ion Collider (RHIC) [4]. In fact, it was the challenge presented by the quark-gluon plasma which motivated the activity leading to the formulation of the KSS bound (1).

Where does the KSS bound (or any other refined bound on the ratio η/s) come from ? It is not clear how to obtain such a bound directly from microscopic physics [3]. Inspection of the Green-Kubo formula [5] which relates the viscosity of a fluid to its fluctuations shows no apparent connection of the viscosity and the entropy density. Such microscopic consideration affords no special status to the ratio η/s.

Where should we look for the physical mechanism which bounds the ratio η/s of viscosity to entropy density ? It is well known that the viscosity coefficient η characterizes the intrinsic ability of a perturbed fluid to relax towards equilibrium [6] [see Eq. (2) below]. The response of a medium to mechanical excitations is characterized by two types of normal modes, corresponding to whether the momentum density fluctuations are transverse or longitudinal to the fluid flow. Transverse fluctuations lead to the shear mode, whereas longitudinal momentum fluctuations lead to the sound mode. These perturbation modes are characterized by distinct dispersion relations which describe the poles positions of the corresponding retarded Green functions [2].

Let us examine the behavior of the shear mode for fluids with zero chemical potential. The Euler identity reads $\epsilon + P = Ts$, where ϵ is the energy density, P is the pressure, T is the temperature, and s is the entropy density of the fluid. The dispersion relation for a shear wave with frequency ω and wave vector $k \equiv 2\pi/\lambda$ is given by [7, 8]:

$$\omega(k)_{\text{shear}} = -i\frac{\eta}{Ts}k^2 + O\Big(\frac{\eta^3 k^4}{s^3 T^3}\Big) \ , \tag{2}$$

where η is the shear viscosity coefficient of the standard first-order hydrodynamics. The correction term becomes small in the $\eta/s \ll 1$ limit, the case of most interest here.

The imaginary part of the dispersion relation entails a damping of the perturbation mode. Its magnitude therefore quantifies the intrinsic ability of a fluid to dissipate perturbations and to approach thermal equilibrium.

It is important to realize that hydrodynamics is actually an effective theory. In the most common applications of hydrodynamics the underlying microscopic theory is a kinetic theory. In these cases the microscopic scale which limits the validity of the effective hydrodynamic description is the mean free path l_{mfp} [7]. In some other cases the underlying microscopic theory is a quantum field theory, which might not necessarily admit a kinetic description. In these cases, the role of the parameter l_{mfp} is played by some typical microscopic scale like the inverse temperature: $l_{\text{mfp}} \sim T^{-1}$. One therefore expects to find a breakdown of the effective hydrodynamic description at spatial and temporal scales of the order of [7]

$$l \sim \tau \sim T^{-1} \ . \tag{3}$$

Below we shall make this statement more accurate.

At this point, it is worth emphasizing that the conjectured KSS bound is based on holographic calculations of the shear viscosity for strongly coupled quantum field theories with gravity duals [1, 2]. These holographic arguments serve to connect quantum field theory with gravity. This fact indicates that a derivation of a KSS-like bound may require use of the elusive theory of quantum gravity. This may seem as bad news for our aspirations to prove (a refined version of) the KSS bound. But one need not loose heart– there is general agreement that black hole entropy reflects some aspect of the quantum theory of gravity [3].

The realization that a black hole is endowed with well-defined entropy $S_{BH} = A/4$, where A is the surface area of the black hole [9], has lead to the formulation of the generalized second law (GSL) of thermodynamics. The GSL is a unique law of physics that bridges thermodynamics and gravity [3, 9]. It asserts that in any interaction of a black hole with an ordinary matter, the sum of the entropies (matter+hole) never decreases. One of the most remarkable predictions of the GSL is the existence of a universal entropy bound [10]. According to this universal bound, the entropy contained in a given volume should be bounded from above:

$$S \leq 2\pi RE \ , \tag{4}$$

where R is the effective radius of the system and E is its total energy.

Furthermore, the generalized second law allows one to derive in a simple way two important new quantum bounds:

- The universal relaxation bound [11, 12]. This bound asserts that the relaxation time of a perturbed thermodynamic system is bounded from below by

  $$\tau \geq 1/\pi T \ , \tag{5}$$

  where T is the temperature of the system. This bound can be regarded as a quantitative formulation of the third law of thermodynamics. One can also write this bound as $\Im\omega \leq \pi T$, where ω is the quasinormal frequency of the perturbation mode.

- A closely related conclusion is that thermodynamics can not be defined on arbitrarily small length scales. The minimal length scale (radius) ℓ for which a consistent thermodynamic description is available is given by $\ell_{min} = 1/2\pi T$ [3, 12].

The longest wavelength which can fit into a space region of effective radius ℓ is $\lambda_{max} = 2\pi\ell$. Thus, the GSL predicts that an effective hydrodynamic description is limited to perturbation modes with wavelengths larger than $2\pi\ell_{min} = T^{-1}$. We note that this exact limit is in accord with the heuristic arguments which lead to the approximate relation (3). The breakdown of the effective hydrodynamic description for perturbation modes with wavenumbers k larger than $2\pi T$ should manifest itself in the hydrodynamic dispersion relation (2). Namely, one should expect to find a violation of the universal relaxation bound (5) by short wavelength perturbations with $k > 2\pi T$.

A lower bound on the ratio η/s can be inferred by substituting $k = 2\pi T$ in the shear dispersion relation (2) and requiring that $\Im\omega \geq \pi T$ for this limiting value of the wavenumber.

As discussed above, the GSL predicts that the effective hydrodynamic description breaks down for short wavelength perturbations with $k \geq 2\pi T$. This should be reflected in the hydrodynamic shear dispersion relation in the form of a violation of the universal relaxation bound (5). Explicitly, these wavenumbers should be characterized by $\Im\omega \geq \pi T$. This physical condition implies the inequality

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \; . \tag{6}$$

We have therefore proved the previously conjectured viscosity-entropy bound.

Summary.— The conjectured bound on viscosity has been the focus of much recent attention. However, despite the flurry of research in this field the origin of the bound has remained somewhat unclear. In this essay we showed that the GSL, a law whose very meaning stems from gravitation, may shed much light on the origins of this mysterious bound. In particular, it was shown that the viscosity bound is actually a direct outcome of the interplay between gravity, quantum theory, and thermodynamics.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); M. Brigante, H. Liu, R. C. Myers, S. Shenker, and S. Yaida, Phys. Rev. Lett. **100**, 191601 (2008).

[2] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001); A. Buchel and J. T. Liu, Phys. Rev. Lett. **93**, 090602 (2004); P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005); D. T. Son and A. O. Starinets, Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007).

[3] I. Fouxon, G. Betschart, and J. D. Bekenstein, Phys. Rev. D **77**, 024016 (2008).

[4] K. Dusling and D. Teaney, Phys. Rev. C **77**, 034905 (2008); A. Adare *et al.* (PHENIX Collaboration), Phys. Rev. Lett. **98**, 172301 (2007); P. Romatschke and U. Romatschke, Phys. Rev. Lett. **99**, 172301 (2007); H. Song and U. W. Heinz, Phys. Lett. B **658**, 279 (2008).

[5] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Butterworth-Heinemann, Oxford, 2000), Vol. II.

[6] P. Danielewicz and M. Gyulassy, Phys. Rev. D **31**, 53 (1985).

[7] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 0804:100 (2008).

[8] M. Natsuume and T. Okamura, Phys. Rev. D **77**, 066014 (2008).

[9] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); J. D. Bekenstein, Phys. Rev. D **9**, 3292 (1974); S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[10] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981); J. D. Bekenstein, Found. Phys. **35**, 1805 (2005).

[11] S. Hod, Phys. Rev. D **75**, 064013 (2007); S. Hod, Class. Quantum Grav. **24**, 4235 (2007).

[12] A. Gruzinov, arXiv:0705.1725; A. Pesci, Class. Quantum Grav. **24**, 6219 (2007).