On vacuum density, the initial singularity and dark energy*

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Abstract

Standard cosmology poses important questions. Apart from its singular origin, early and late accelerations are required by observations. Vacuum energy is a possible way to resolve some of these questions. Its density in an early de Sitter phase was estimated to be proportional to H^4, while it was suggested that QCD induces a term proportional to H at late times. These results have been employed in models which are non-singular and inflationary at early times and accelerating at late times. Here they are modeled in terms of scalar fields. At early times the spectrum of perturbations is scale-dependent, implying that slow-roll inflation is required after the primordial one. At late times, the scalar-field presents a mass of the order of the Hubble scale.

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Recent years have witnessed a tremendous accumulation of high resolution data in cosmology. This has led to the so called standard model which poses a number of important questions. Apart from having a singular origin, at which laws of physics break down, it also possesses accelerating phases at early and late times. This has resulted in numerous attempts at generalising the gravitational and energy sectors of GR in order to account for these features. Interestingly all viable models have proved to be very close to the ΛCDM.

An important ingredient expected from quantum considerations is the vacuum energy. In particular it has been thought to be responsible for the origin of the cosmological parameter Λ. As is well known, it is not easy to obtain the vacuum density in curved spacetimes, even in the simple case of a scalar field. An exception is the case of fields with conformal invariance in de Sitter spacetime, for which the renormalized vacuum density has been found to be proportional to H^4, where H is the Hubble parameter [1–3]. In a general FLRW background, some ambiguities can be fixed by imposing the conservation of the vacuum energy-momentum tensor [3]. This allows a general expression for the vacuum density to be derived, which was in fact used by Starobinsky in his pioneering inflationary model [4].

Recently the de Sitter ansatz $\Lambda \propto H^4$ was employed in a quasi-de Sitter setting in order to construct a non-singular scenario, with a phase transition between a past-eternal de Sitter phase and a radiation-dominated epoch [5]. An underlying assumption was that a dynamical vacuum in principle interacts with matter, since only the conservation of the total energy-momentum tensor is implied by Einstein equations. Thus in this model the decaying vacuum acts as a source of relativistic matter during the expansion.

On the other hand, it has been suggested that the QCD condensate induces a vacuum density proportional to H at late times [6, 7]. This has also been employed to construct models to account for the late acceleration of the universe.

In this paper we formulate these ansatz in terms of self-interacting scalar fields in a spatially flat FLRW spacetime. In this framework the Lagrangian is given by

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right], \tag{1}$$

with corresponding field equations

$$3H^2 = V + 2H'^2, \tag{2}$$

$$\dot{\phi} = -2H'. \tag{3}$$

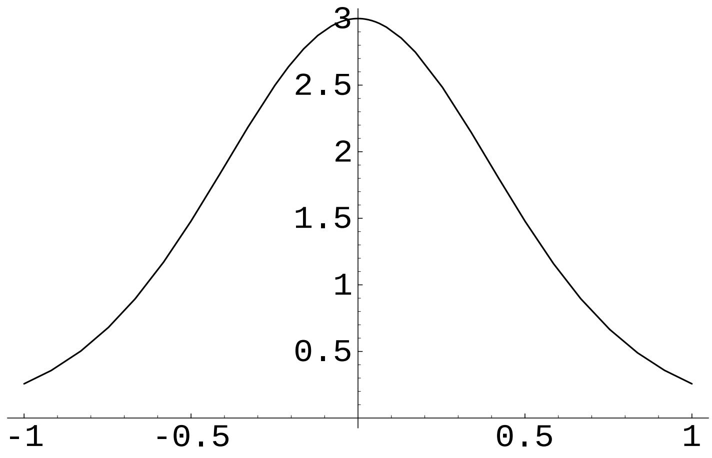

FIG. 1: The scalar field potential as a function of ϕ ($\sigma = 1$).

Here the prime and the dot denote derivatives with respect to ϕ and the cosmological time t, respectively.

We begin with the early phase and consider a potential proportional to H^4, which has a maximum in the asymptotic de Sitter limit. The energy density and pressure of this field are given by $\rho = V + \dot{\phi}^2/2$ and $p = -V + \dot{\phi}^2/2$, respectively. Therefore, we can interpret the material content as formed by a vacuum term with density V and pressure $-V$, plus a stiff fluid with density and pressure equal to $\dot{\phi}^2/2$. As the transition from the de Sitter phase takes place, with the field rolling down the potential, the energy stored in the vacuum term is transferred to the stiff component, and eventually converted into matter fields by a suitable coupling, as in the usual reheating mechanism.

With the ansatz

$$V = 3\sigma H^4, \tag{4}$$

where σ is a positive constant, it is straightforward, using Eqs. (2)-(3), to find the solution

$$H = \frac{2\, e^{\sqrt{3/2}\phi}}{1 + \sigma e^{\sqrt{6}\phi}}. \tag{5}$$

The corresponding potential, with $\sigma = 1$, is shown in Figure 1.

Also, using $\dot{H} = H'\dot{\phi}$ in Eqs. (2)-(3), we obtain an evolution equation for H,

$$\dot{H} + 3H^2 - 3\sigma H^4 = 0. \tag{6}$$

This equation has the equilibrium point $\sigma H^2 = 1$, which corresponds to a de Sitter universe with energy scale $\sigma^{-1/2}$ in Planck units. This solution is, however, unstable, as indicated by

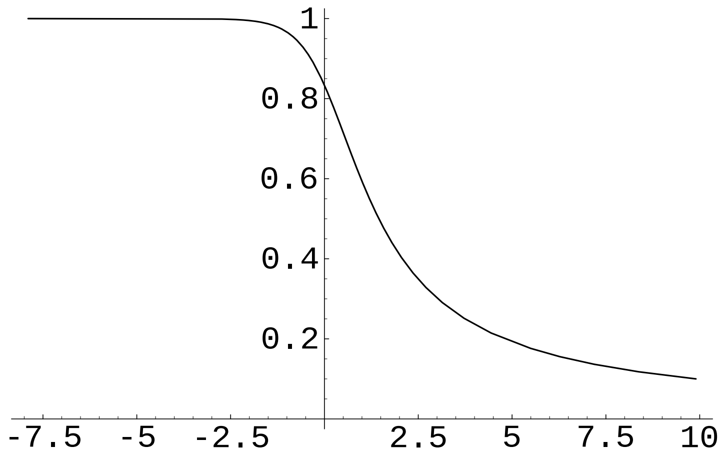

FIG. 2: The Hubble parameter as a function of time ($\sigma = 1$).

the solution

$$\tilde{t} = \frac{1}{\tilde{H}} - \tanh^{-1} \tilde{H}, \tag{7}$$

where we have introduced the re-scaled quantities $\tilde{H} = \sqrt{\sigma} H$ and $\tilde{t} = 3t/\sqrt{\sigma}$, and have conveniently chosen the integration constant.

The time evolution of $\tilde{H}$ is depicted in Figure 2. As $t \to -\infty$ the solution approaches the de Sitter point, with $\sigma H^2 = 1$, as expected. The solution remains quasi-de Sitter during an indefinitely long time. Then, around $t = 0$, it goes through a phase transition, with H and V decaying very fast. It is worth noting that, for $\sigma > 1$, $H < 1$ during the entire evolution, thus we never enter a trans-Planckian regime, where the semi-classical, one-loop derivation of the vacuum density would not be valid.

This model has a number of appealing features. In addition to being non-singular, the presence of a quasi-de Sitter phase solves some of the problems usually addressed by inflation, such as the horizon and flatness problems. However, the most important outcome of an inflationary phase is to produce a scale invariant spectrum of primordial perturbations. To be viable as an inflationary scenario, it is therefore important to check whether the above model can also produce such a scale-invariant spectrum.

The evolution equation for perturbations in the scalar field is given by [8]

$$\delta\phi^{**} + 2aH\delta\phi^{*} + (k^2 + a^2 V'')\delta\phi = 0, \tag{8}$$

where a is the scale factor, k is the co-moving wave number of a given mode, and * denotes a derivative with respect to the conformal time η.

From (4), (5) and (3) it is possible to obtain

$$V'' = 18\sigma H^4(4 - 5\sigma H^2), \tag{9}$$

$$\epsilon = 3(1 - \sigma H^2), \tag{10}$$

with $\epsilon = d(H^{-1})/dt$. In the quasi-de Sitter phase we have $\sigma H^2 \approx 1$ and, hence, $V'' \approx -18H^2$ and $\epsilon \ll 1$. By using $Ha \approx -1/\eta$ and introducing $\delta\tilde{\phi} = a\delta\phi$, (8) can be rewritten as

$$\delta\tilde{\phi}^{**} + \left(k^2 - \frac{20}{\eta^2} + \frac{30\epsilon}{\eta^2}\right)\delta\tilde{\phi} = 0. \tag{11}$$

Neglecting the term in ϵ, the appropriately normalized solution has the form

$$\delta\tilde{\phi} \approx \frac{\sqrt{-\pi\eta}}{2}\, H^{(1)}_{\frac{9}{2}}[-k\eta], \tag{12}$$

where $H^{(1)}_n$ is the Hankel function of the first kind. For $k \to \infty$, this solution reduces to $e^{-ik\eta}/\sqrt{2k}$, as expected. On the other hand, for $k\eta = -1$, when the mode crosses the horizon, we have $|\delta\tilde{\phi}| \approx 10^2/\sqrt{2k}$.

The power spectrum of scalar perturbations in the metric is given by [8]

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\phi^*}\right)^2 |\delta\phi|^2_{k\eta=-1}\,. \tag{13}$$

Using (12) and $(aH/\phi^*)^2 = (2\epsilon)^{-1}$, we have

$$k^3 P_\Psi \approx \frac{10^4\, H^2}{9\epsilon}, \tag{14}$$

with the r.h.s. evaluated at the horizon crossing. Apart from the factor of 10^4, this is the same expression we find in slow-roll inflation. The power spectrum of tensor modes is the same, and hence the extra factor 10^4 leads to a stronger suppression of primordial gravitational waves. Nevertheless, ϵ here has a strong dependence on k, which makes the spectrum scale-dependent. Indeed, from (10) it is possible to show that, for $\sigma H^2 \approx 1$, $\epsilon \propto \eta^{-6}$. Therefore, at the horizon crossing we have $\epsilon \propto k^6$, leading to a scalar spectral index $n - 1 \approx -6$. This is related to the fact that we do not have a slow-roll potential, since from (4) and (9) we have, for $\sigma H^2 \approx 1$, $V''/V = -6$. Therefore, this scenario cannot by itself produce a scale-invariant spectrum. To achieve this, the phase transition described here must be followed by a usual inflationary epoch.

Let us now discuss the potential role of the vacuum energy at late times. The free-field vacuum density, of order H^4, is presently too small to be identified with the dark energy. On the other hand, any contribution proportional to H^2 may be absorbed by the gravitational constant in Einstein equations [9]. It is usually believed that, apart from terms in H^2 and H^4, the only remaining contribution is a free constant, which may be absorbed by a cosmological constant. Nevertheless, it has recently been claimed that the introduction of interactions may change this picture [6, 7]. In particular, the QCD condensate leads to a vacuum density of order $m^3 H$, where $m \approx 150$ MeV is the energy scale of the QCD phase transition. Although not conclusive, this result gives the correct order of magnitude for the cosmological term, $\Lambda \approx m^6$, and, as a byproduct, the Dirac large number coincidence $H \approx m^3$.

It is not clear whether this linear relation is valid only in the de Sitter limit or can also be considered dynamical near this limit. Again, in order to ensure the conservation of total energy, matter production would be required. This scenario was studied recently, by confronting its predictions with the current observations. A joint analysis of SNe Ia, baryonic acoustic oscillations and the position of the first peak of CMB anisotropy spectrum produced a good concordance [10]. However, the production of dark matter leads to a power suppression in the mass spectrum, which seems to rule out this scenario [11]. This difficulty may be overcome if we avoid matter production by associating the varying Λ with a quintessence field, as was done in the early-time limit above. A full study, which also includes matter, is in progress. Here we shall consider the simpler case where only the quintessence field is present, which is reasonable very near the de Sitter limit.

With the ansatz $V = 3\sigma H$, where now $\sigma \approx m^3$, Eqs. (2)-(3) have the solution

$$V = \frac{3\sigma \left(e^{\sqrt{3/2}\phi} + \sigma \right)^2}{4 e^{\sqrt{3/2}\phi}}, \tag{15}$$

$$H = \frac{\sigma e^{3\sigma t}}{e^{3\sigma t} - 1}. \tag{16}$$

As can be seen, as $t \to \infty$, $H \to \sigma$ as expected.

The potential possesses a minimum at $\phi_0 = 2 \ln \sigma / \sqrt{6}$, corresponding to the de Sitter limit. Around this point it can be expanded as

$$V \approx V_0 + \frac{1}{2} M^2 \left(\phi - \phi_0 \right)^2, \tag{17}$$

where $V_0 = 3\sigma^2$ and $M = 3\sigma/2$. The mass term is worthy of note. In the de Sitter limit we have $\sigma = H = \sqrt{\Lambda/3}$. Therefore, the mass of the quintessence field is given by

$$M = \frac{3}{2}\sqrt{\Lambda/3}. \tag{18}$$

This is the mass expected for elementary degrees of freedom in the context of the holographic conjecture. Indeed, recalling the observable mass in the universe, $E \approx \rho/H^3 \approx 1/\sqrt{\Lambda}$, and the entropy associated with the de Sitter horizon, $N \approx 1/H^2 \approx 1/\Lambda$, we obtain $M \approx E/N \approx \sqrt{\Lambda}$ for the mass of each degree of freedom.

In conclusion, we note that the vacuum energy, usually considered a problem in quantum field theories and cosmology, may actually shed new light on other fundamental problems, such as the initial singularity and the nature of dark energy.

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