

### GRAVITATIONAL WAVES

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#### Abstract

Two methods are proposed for detection of gravitational waves. One is based on the fact that the space derivatives of the fields of the waves give rise to motion of a system of masses relative to each other. Mechanical resonance or excitation of acoustic vibrations then enables detection to be accomplished. The second method makes direct use of the strains set up by the space derivatives of the wave field. These strains may result in electric polarization in consequence of the piezo electric effect. Electrical resonance is employed and acoustic vibrations are suppressed. Mathematical analysis of the limitations is given. Under certain conditions at least one component of the Riemann tensor can be measured. These methods provide a much better way of detecting interstellar gravitational waves than interpretation of known astronomical anomalies.

The generation of gravitational waves in the laboratory is discussed. A method is proposed and analyzed which appears to give a seventeen order increase in radiation over that of a spinning rod.

## Part I

### Detection of Gravitational Waves

Suppose we have two masses  $m_1$  and  $m_2$ , separated by a spring (Figure 1). Then, if a time dependent gravitational

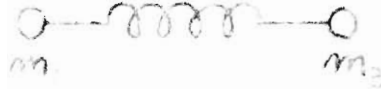


Figure 1

field is applied which is uniform over the apparatus, every particle will have the same acceleration at any given time. There is no relative motion of  $m_1$  and  $m_2$ . The situation is different if a gravitational wave is incident. The wave intensity is not uniform in space and the phase at  $m_1$  is in general different from the phase at  $m_2$ . Relative motion of  $m_2$  is now possible with respect to  $m_1$ . Therefore energy may now be abstracted from the wave.

Alternatively a solid block of crystalline material may be employed (Figure 2)



Figure 2

The gravitational wave sets up compressional or shear mode oscillations. At resonance these may have large amplitude and transfer energy to other devices.

We employ Einstein's General Theory of Relativity, in the weak field approximation. The metric tensor is given

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (1)$$

$\eta_{\mu\nu}$  is the Lorentz metric and  $h_{\mu\nu}$  is first order quantity. The force per unit mass is obtained from the Christoffel symbol  $\Gamma^i_{\mu\nu}$ , where to a first approximation

$$\Gamma^i_{00} = \frac{1}{2} \eta^{ij} h_{00,j} \quad (2)$$

The force per unit mass in the  $i^{\text{th}}$  direction will be denoted by  $f^i$ , and

$$\ddot{x}^i = c^2 \Gamma^i_{00} \quad (3)$$

In (3)  $c$  is the speed of light. (3) follows directly from the geodesic equation

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

For gravitational waves radiated by an oscillating linear mass quadrupole the relation between wave power flow per unit area  $\tau_{or}$  and the squared force per unit mass  $f^2$  is given<sup>1</sup> by

<sup>1</sup> Rosen and Sherrill, Reviews of Modern Physics 29, 429, July, 1957.

$$|f|^2 = \frac{176\pi G \dot{E}_{or}}{c} \quad (4)$$

The relation (4) assumes that the frame is at rest relative to the center of mass of the oscillating quadrupole.  $G$  is the constant of gravitation, and  $\dot{E}_{or}$  is the component of the stress energy pseudo tensor representing power flow per unit area.

Consider now the oscillator of Figure 1, driven by a sinusoidal gravitational wave with force

$$f = f_0 e^{i(\omega t - \beta \cdot \bar{x})}$$

Let the position vectors of the masses be  $\bar{r}_1$  and  $\bar{r}_2$  and let their displacements be  $\bar{\xi}_1$  and  $\bar{\xi}_2$ . The equations of motion are

$$m_1 \frac{d^2 \bar{\xi}_1}{dt^2} + D \frac{d}{dt} (\bar{\xi}_1 - \bar{\xi}_2) + k (\bar{\xi}_1 - \bar{\xi}_2) = \bar{f}_1 m_1 e^{i(\omega t - \beta \cdot \bar{r}_1)} \quad (5)$$

$$m_2 \frac{d^2 \bar{\xi}_2}{dt^2} + D \frac{d}{dt} (\bar{\xi}_2 - \bar{\xi}_1) + k (\bar{\xi}_2 - \bar{\xi}_1) = \bar{f}_2 m_2 e^{i(\omega t - \beta \cdot \bar{r}_2)} \quad (6)$$

In (6)  $D$  is a dissipation factor and  $k$  is the spring constant.

We subtract (6) from (5) and make the substitution  $\bar{\xi} = \bar{\xi}_1 - \bar{\xi}_2$

Let  $m_1 = m_2 = m$ , let  $2D = R$  and  $2k = K$ . We then obtain

$$m \frac{d^2 \xi}{dt^2} + k \xi = \gamma_0 e^{i(\omega t - \beta \bar{r}_i)} \quad \left[ \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right] \quad (7)$$

The steady state solution of (7) is  $\xi = \xi_0 e^{i\omega t}$  with

$$\xi_0 = \frac{\gamma_0 m \left[ \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right] e^{-i\beta \bar{r}_i}}{(-\omega^2 m + i\omega R + k)} \quad (8)$$

The total mechanical resistance is  $R = R_i + R_e$ , where  $R_i$  is the internal resistance and  $R_e$  is the external resistance to which the apparatus is coupled. At resonance the denominator of (8) reduces to  $i\omega R$  and the power which may be transferred to an absorber of energy is

$$\frac{1}{2} \omega^2 R_e |\xi_0|^2 = \frac{c |\gamma_0|^2 \omega^2 \left( \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right)^2 m^2 R_e}{(R_e + R_i)} \quad (9)$$

(9) is a maximum when  $R_e = R_i$ . Let  $P_M$  be the maximum power, then the use of expression (4) gives

$$P_M = \frac{88\pi G m^2 \omega^2 \left( \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right)^2 t_{0,2}}{c R_i} \quad (10)$$

(10) may be written in terms of the mechanical  $Q = \frac{\omega m}{R}$ , as

$$P_M = \frac{88\pi G \omega^2 \left( \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right)^2 m Q}{\omega c} t_{0,2} \quad (11)$$

The absorption cross section implied by (11) is denoted by  $S_A$  where

$$S_A = \frac{88\pi G m Q \left( \frac{\beta \cdot (\bar{r}_1 - \bar{r}_2)}{r} \right)^2}{\omega c} \quad (12)$$

Detailed analysis shows that an upper limit is

reached by (12) when all internal damping is absent and the external resistance is equal to the effect of the gravitational radiation damping as the antenna oscillates. In this case we have

$$(S_A)_{\text{Radiation Damped}} = \frac{17 \lambda^2}{4 \pi} \quad (12A)$$

where  $\lambda$  is a gravitational wave wavelength, related to  $\beta$  by  $\beta = \frac{2 \pi}{\lambda}$ . The condition which led to (12A) is realizable for a radio antenna but not for a gravitational wave antenna because the internal damping is always many orders greater than the gravitational wave damping. The  $Q$  associated with gravitational wave damping exceeds  $10^{20}$ . Practical devices will have a  $Q < 10^6$ , in which case the absorption cross section for a practical antenna may approach  $10^{-12} \text{ cm}^2$ . While this is very small it appears good enough to try certain experiments to be outlined later.

If a continuous spectrum of radiation is incident on the antenna the power absorbed is given by

$$P_A = \frac{1}{T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{4 \pi^2 \text{Re } \omega \omega' f_0(\omega) f_0^*(\omega') e^{i(\omega - \omega')t}}{z(-\omega^2 m + i\omega R + K)(-\omega'^2 m + i\omega' R + K)} d\omega d\omega' dt \quad (13)$$

Evaluating (13) by contour integration gives

$$P_A = 4 \pi^2 G m c^{-1} \text{mm}^2 \left( \frac{\bar{\beta}(\omega_0) \cdot [\bar{R}_1 - \bar{R}_2]}{z} \right) t_{02}(\omega_0) \quad (14)$$

In (14)  $\tau_{or}(\omega_0)$  is the power spectrum of the incident gravitational flux in the vicinity of the apparatus resonant frequency  $\omega_0$ .

We consider now the excitation of a solid block of material (Figure 2). To simplify the discussion of relative motions we imagine that one plane of the block of material falls freely with the incident wave and we analyze accelerations relative to this plane. The equations of motion are

$$Y \frac{\partial^2 \xi}{\partial x^2} + \rho \frac{\partial \xi}{\partial t} + R \frac{\partial^2 \xi}{\partial t^2} = \frac{\rho}{c^2} \omega_0^2 \left[ e^{-i\omega_0 x} - 1 \right] \quad (15)$$

In (15)  $Y$  is the appropriate elastic modulus,  $\rho$  is the density,  $R$  is a volume resistivity. The term  $R \frac{\partial^2 \xi}{\partial t^2}$  takes losses into account. The right hand side of (15) has the factor  $e^{-i\omega_0 x}$

because accelerations are being measured relative to a plane that falls with the apparatus. Let  $v_s$  be the sound velocity,

let sound wavelength be  $\lambda_s$ , let  $k_s = \frac{2\pi}{\lambda_s}$ , let  $\alpha = \frac{R}{\rho v_s}$ ; then

we define  $\gamma = \alpha + i k_s$  and to a good approximation the solution of (15) may be written

$$\xi = \left[ A \sinh \gamma x + \frac{1}{\gamma} \left( 1 - e^{-\gamma x} \right) \right] e^{i\omega_0 t} \quad (16)$$

16)  $\beta_x$  is the component of the gravitational wave phase constant  $\beta$  in the direction of propagation of the acoustic (longitudinal) waves, and  $\xi$  is the acoustic wave displacement. We now discuss two cases. In the first case

we encourage acoustic wave generation and arrange to have  $A$  in (16) as large as possible. Making use of the boundary condition that the acoustic pressure  $Y \frac{dy}{dx}$  vanishes at the ends leads (for small  $R$ ) to

$$A = \frac{F_0 \lambda_s \left( \frac{\beta_2}{\beta} \right)}{\omega^2 \lambda \left( \sin k_s l + \cos k_s l \right)} \quad (17)$$

In (17)  $\lambda$  is half the length of the block. The largest value of (17) is reached at the first acoustic wave resonance,  $k_s l = \frac{1}{2}$ . The analysis of equations (5) through (14) must be replaced by the acoustic wave treatment when the length of the spring approaches half an acoustic wavelength. The spacing of the masses  $m_1$  and  $m_2$  corresponding roughly to half an acoustic wavelength gives best performance. This is an important limitation because the gravitational wavelength is much longer than the acoustic wavelength, and the effective phase shift difference which drives the oscillator is very small. Also the value of  $m$  for best performance is

$$m_1 = \frac{\rho A \lambda_s}{2\pi} \quad (18)$$

In (18)  $A$  is the cross sectional area of the block. The acoustic wavelength limitations can be overcome and improved operation made possible under many conditions in the following way. Choose the length or other dimensions of the block of crystal so that acoustic resonance vibrations

are not excited. Then the first term of equation (16) is not important. The wave will however induce a strain  $\epsilon_{xx}$  given by

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{c c'}{c^2} e^{i(\omega t - c x/c)} \quad (19)$$

It is instructive to return, for a moment, to fundamentals. Expression (19) may be written in terms of the Christoffel symbols as

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{c c'}{c^2} \frac{\partial^2 \xi}{\partial x^2} \quad (20)$$

More generally the strain tensor  $\epsilon_{ij}$  will be given by

$$\epsilon_{ij} = \frac{c c'}{c^2} \left[ \frac{\partial^2 \xi}{\partial x^i \partial x^j} + \frac{\partial^2 \xi}{\partial x^i \partial x^j} \right] \quad (21)$$

The derivatives of the Christoffel symbols cannot be transformed away over a region. The strains induced by the gravitational wave are a real effect. We emphasize that these strains are due to relative motion of masses and are very much greater than the effects due to changes in distance between points resulting from the time dependent metric alone.

In a piezo electric crystal a strain results in an electric polarization  $P_i$  given by

$$P_i = \epsilon_{ki} E_k \quad (22)$$

In (22)  $E_i^{kl}$  is the piezo electric stress tensor. The electric polarization in (22) gives rise to an electric field over the crystal. Since acoustic vibrations are assumed small,  $P_i$  does not change its sign every half acoustic wavelength. It can integrate to a large value, giving a terminal voltage which can transmit power to an amplifier of weak signals. For simplicity we assume that the crystal is polarized in one direction only. Let  $V$  be the volume of the block of crystalline material. Let  $Q_e$  be the electrical circuit  $Q$  defined by

$$Q_e = \frac{\omega (\text{Stored Energy})}{\text{Power Dissipated}} \quad (23)$$

The power which the detector can deliver to an associated electric circuit is readily calculated using (21), (4), and (23). We assume that the piezo electric material is barium titanate. In this case the absorbed power  $P_A$  is given by

$$P_A = 4.5 \times 10^{-22} \omega^2 V Q_e t_{or} \text{ ergs per second} \quad (24)$$

In (24)  $\omega$  is again the angular frequency and  $t_{or}$  is the incident gravitational flux in ergs per square centimeter per second. For a continuous spectrum of gravitational radiation with power spectrum function  $t_{or}(\omega)$  the power absorbed by an electric circuit of resonant frequency  $\omega_0$  may be calculated using the contour integral method used earlier and is (for barium titanate)

$$P_{N1} = + \dots \dots \dots (25)$$

In order to detect the absorbed power given by (24) (25), (11) and (14), amplification by electrical means is necessary and the electrical fluctuation noise in the antenna and amplifier must be considered. For synchronous detection of the sinusoidal waves the power output of the detector (expressions (10) and (24)) must exceed the noise power  $P_{N1}$  given by<sup>2</sup>

$$P_{N1} = \frac{N h \omega}{\dots} \dots \dots (26)$$

In (26)  $h$  is Planck's constant divided by  $2 \pi$ ,  $k$  is Boltzmann's constant,  $T$  is the gravitational antenna temperature and  $N$  is the noise factor of the receiver which is expected to be less than 25, and more than 1.  $\tau$  is the averaging time. For a continuous spectrum of gravitational radiation the power (expressions (14) and (25)) delivered by the detector must exceed the noise power  $P_{N2}$  given by<sup>2</sup>

$$P_{N2} = \frac{h \omega}{\dots} \dots \dots (27)$$

<sup>2</sup> R. H. Dicke Rev. Sci. Inst. 17, 268, 1946

We are planning experiments to search for interstellar gravitational radiation using both methods described here. For the first method we use the earth itself as the block of material constituting the antenna. The earth's normal modes (about 1 cycle per hour) are excited by incident gravitational waves. This procedure is limited by the relatively low  $Q$  of the earth and the high noise temperature of the earth's core. The apparatus of Figure 3 is employed in the second method, in which acoustic oscillations of the crystal block are suppressed. Search at frequencies  $\sim 10^3$  cycles per second is planned. The earth rotates the apparatus. If radiation is incident from a particular direction it may be observed from the diurnal change in amplifier noise output. The arrangement of Figure 4 will be used to search for isotropic gravitational radiation.

Use of the analysis given here predicts that gravitational flux with a power spectrum  $t_{gr}(\omega) \approx 10^{-4}$  ergs/cm<sup>2</sup> second cycle should be detectable. This is of the order predicted by certain cosmological theories.

The velocity of propagation of gravitation waves must be the speed of light. This follows from the agreement with experiment of the calculated advance in the perihelion of mercury, which employs  $c$  for the propagation velocity of gravitational interactions. The apparatus described here has a known resonant frequency, therefore it gives not only

Shield

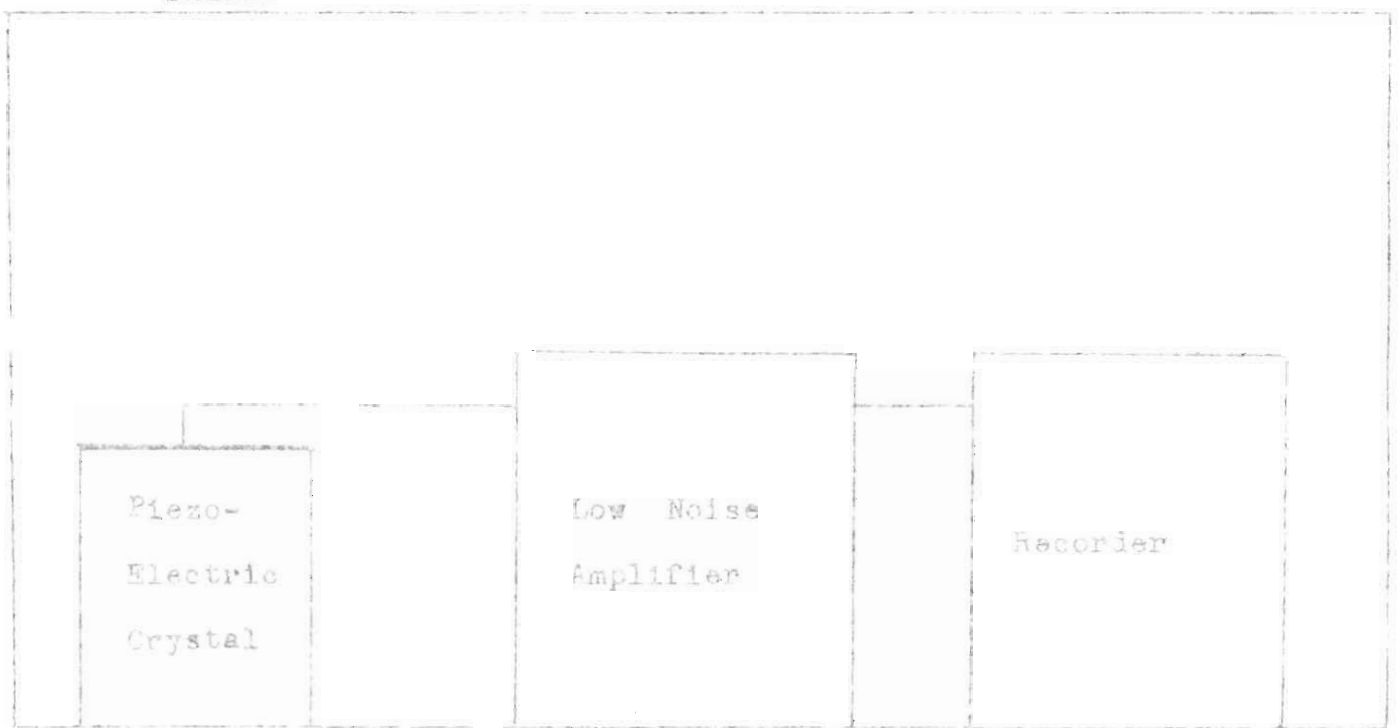


Figure 3

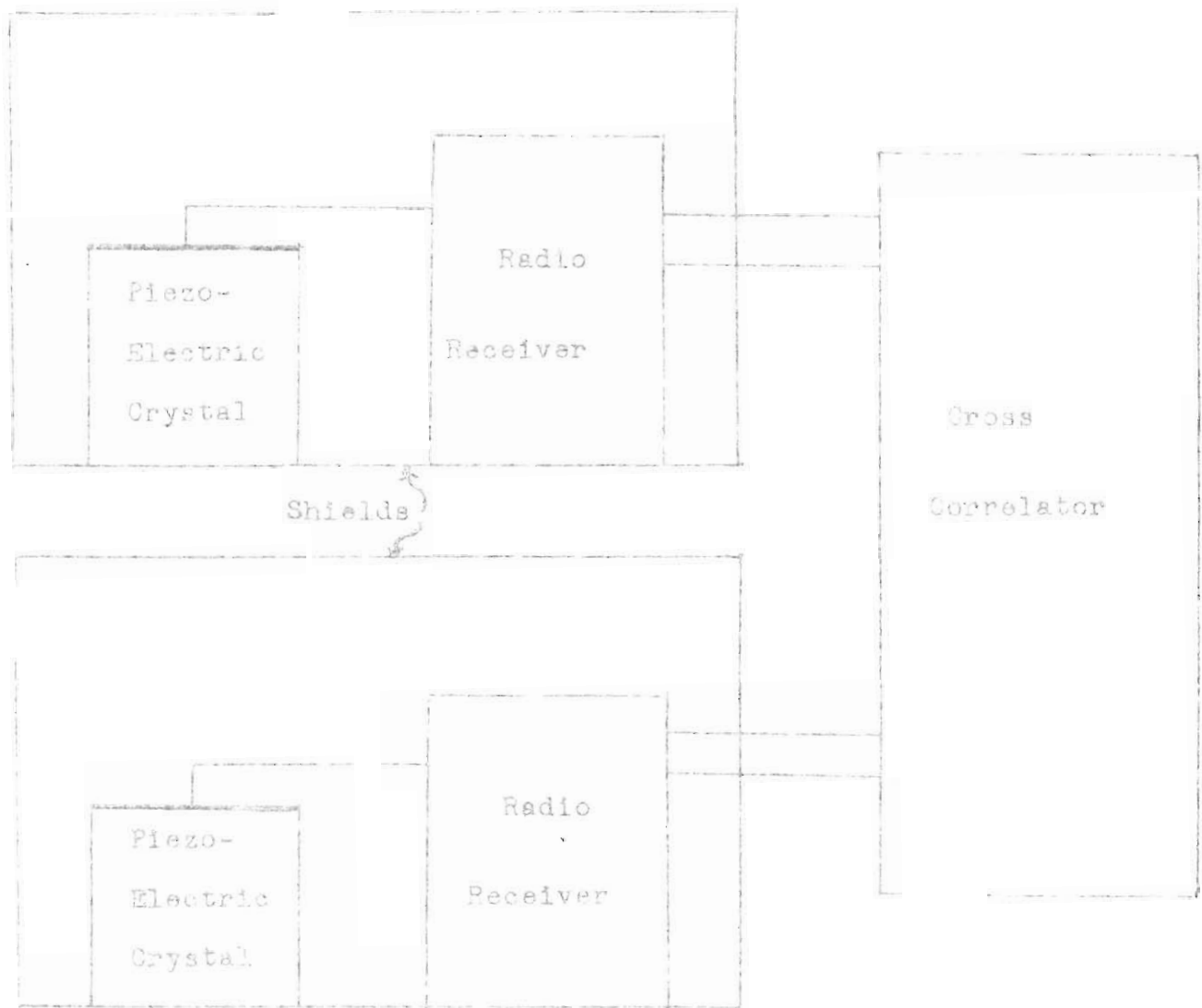


Figure 4

$h_{\mu\nu}$  but also the second space and time derivatives of  $h_{\mu\nu}$ , since the velocity is known to be  $c$ . If the force vector of the wave is found to be linearly polarized in the direction  $x^k$  the measurements allow direct determination of the component of the Riemann tensor  $R_{\alpha\beta\gamma\delta}$ .

Dirac has suggested that astronomical anomalies might be correlated with effects of gravitational radiation. We have analyzed this possibility. If a gravitational wave is incident on the earth the phase retardation effects give rise to torques. For a continuous spectrum of gravitational radiation this will cause an irregular flutter of the earth's rotation period. Detailed mathematical analysis gives the formula (for a body rotating with angular velocity  $\omega$ )

$$\frac{\overline{I^2}}{I_0^2} = \frac{25\pi G t_{or}}{\omega^2 c^3} \quad (28)$$

In (28)  $\overline{I^2}$  is the mean squared fluctuation in the earth's angular momentum,  $I_0$  is the angular momentum of rotation,  $t_{or}$  is the total gravitational wave flux in ergs per square centimeter per second. If we arbitrarily assume that all the earth's rotational anomalies are due to gravitational waves,  $t_{or}$  is calculated to be  $5 \times 10^8$  ergs per square centimeter per second. It is clear from this that the earth's rotation is a poor detector. The other astronomical anomalies lead to larger figures.

## Generation of Gravitational Waves

It would of course be very desirable if gravitational waves could be generated in the laboratory. For a spinning rod Einstein calculated the rate of radiation to be

$$P_R = 1.73 \times 10^{-54} I^2 \omega^6 \text{ ergs per second.} \quad (29)$$

Here  $I$  is the moment of inertia and  $\omega$  is the angular frequency. If we make  $\omega$  so large that the rod is about to break up we find that the length of the rod is related to the wavelength of sound  $\lambda_s$  at the angular frequency of rupture by

$$l = \lambda_s \sqrt{\delta} \quad (30)$$

In (30)  $\delta$  is the maximum allowed strain for the given material, about  $10^{-30}$  ergs per second can thus be radiated by a one meter rod.

In the linear approximation the solutions of Einstein's field equations are (at a point  $r$  centimeters from the radiator),

$$h_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} h = \frac{4G}{c^4 r} \int (T_{\mu\nu})_{\text{RETARDED}} d^3x \quad (31)$$

In (31)  $T_{\mu\nu}$  is the stress energy tensor. (31) suggests that an oscillating stress energy tensor is one way to generate a gravitational wave. This can be accomplished by electrically

driving a piezo electric crystal. Again we have a choice of either making use of acoustic resonance or suppressing it. If acoustic resonance is employed the length of each resonator is limited to a half acoustic wavelength for best results, and this results in many small resonators, properly phased. It appears better to suppress the resonance vibrations and create, by the piezo electric effect, mechanical stresses which are almost uniform over the crystal. Single large crystals may then be used. These stresses can be made to oscillate harmonically with time if the crystal is driven by a powerful vacuum tube radio-frequency oscillator. Analysis shows that the optimum crystal size is a cube each side of which is a half gravitational wavelength long. The amplitude of the induced stresses is limited by the tensile strength of the crystal. Expression (31) enables us to calculate the gravitational field radiated by the oscillating electric field induced stresses. The stress energy pseudo tensor may then be employed to directly calculate the radiated power  $P_R$  given by (for one crystal)

$$P_R = \frac{G P_{MAX}^2 \lambda^4 \pi^2}{20 c^3} \quad (32)$$

In (32)  $P_{MAX}$  is the tensile strength in dynes per square centimeter, again  $\lambda$  is the gravitational wave wavelength and  $c$  is the speed of light.

For example waves one meter long could be radiated by a crystal fifty centimeters on a side. If driven just below the breaking point each crystal would radiate  $10^{-13}$  ergs each second. This is about  $10^3$  gravitons per second (at  $\omega = 2\pi \times 10^8$ ). Single detectors of the type considered earlier can detect a flux of about  $10^{-3}$  ergs per square centimeter per second. It is apparent that a substantial gap still exists between what can be generated and what can be detected in a small laboratory. Large numbers of radiating elements and a complex detection array can narrow this gap. The electrical power required to drive each crystal unit is large, about  $10^8$  watts. This is not beyond the range of existing vacuum tubes, or means for cooling the crystal.

### Conclusion

The detectors which have been proposed are sufficiently good to search for interstellar gravitational radiation. Further advances are required in order to generate and detect gravitational waves in the laboratory.